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Chapter 17

The Transcendental Aesthetic of Space

Of all things that are, the greatest is place, for it holds all things; the swiftest is mind, for it speeds everywhere; the strongest, necessity, for it masters all; the wisest, time, for it brings everything to light.

Thales

§ 1. The Idea of Space

Possibly nothing in Kant's philosophy has left more room for confusion and debate than his writings on the pure intuition of space. In no small part this is due to Kant's aggravatingly brief discussion of what was nothing less than a radical and revolutionary idea in philosophy. But in part it is also due to a pervasive tendency to admix the idea of space with that of geometry, and to a seeming obviousness of what is meant by the term "space." For most of us, "space" taken as an object means "physical space," and there would seem to be no difficulty with this idea until we are asked to define what we mean by it. The idea of space seems to the adult mind to be both primitive and obvious. The meaning of the word "space" is usually taken to be so self-evident that mathematics, physics, and engineering textbooks do not bother to define it, even as they introduce adjectives to distinguish different technical species of space such as "topological" space, "metric" space, "Hilbert" space, "Fock" space, "state" space, "input" space, "solution" space, & etc. in a list of ever-growing length. But what is the "space" all these various adjectives modify and specify? Is there one of these more privileged than the others so that they are mere species under the genus of this space *per se*? That question has dogged philosophers since before the time of Plato and Aristotle, set Newton and Leibniz at odds with each other, and hinders the efforts of physicists to clearly explain to the rest of us what they mean when they speak of space as something without boundaries which is at the same time "expanding." Space has been held by some to be a thing, and by others to be no-thing but merely a description of relationships among physical things. Einstein once remarked, "Space is not a thing," yet the relativity theory is said to regard space as "curved" by the presence of gravitating masses. If space is not a thing, what is it that is said to be curved?

To put it briefly, the key issue we must explore in this Chapter is, "What does 'space' mean?" We begin with the most common usages. The dictionary lists no fewer than twelve definitions of the word "space":

> **space**, *n*., [OFr. *espace*; L. *spatium*, space, from *spatiari*, to wander.]
> 1. distance extending without limit in all directions; that which is thought of as a boundless, continuous expanse extending in all directions or in three dimensions, within which all material things are contained.
> 2. distance, interval, or area between or within things; extent; room; as, leave a wide *space* between the rows.
> 3. (enough) area or room for some purpose; as, we couldn't find a parking *space*.
> 4. reserved accommodations, as on a train or ship.
> 5. interval or length of time; as, too short a *space* between arrival and departure.
> 6. the universe outside the earth's atmosphere; in full, *outer space*.
> 7. in music, an open place between the lines of the staff.
> 8. in printing, any blank piece of type metal used to separate characters, etc.
> 9. in telegraphy, an interval when the key is open, or not in contact, during the sending of a message.
> 10. time allotted or available for something. [Obs.]
> 11. a short time; a while. [Rare.]
> 12. a path. [Obs.]

Most people most of the time use definitions 1 or 2 when employing the word "space," and use definition 6, perhaps extended a bit so as to include the earth, when talking about "space itself as a thing." It is easy to see definitions 3, 4, 7, 8, and 12 as analogies to one or the other of these first two definitions. Definitions 5, 9, 10, and 11 also follow as analogies from the practice of representing time by means of a geometric line (a "time line"). The idea of "space" as a distance, interval, area, or volume ties "space" to geometry as the mathematical representation for measuring (quantifying) distance, interval, area, or volume. Geometry, we recall, is a word stemming from the Greek for "I measure the earth." Our modern ideas of "space" owe a great deal to our Greek heritage, and we will begin our exploration here with that Greek heritage.

§ 1.1 The Greek Ideas of "Space", "Place", and "Void"

Plato, with his penchant for less-than-precise descriptions, regarded space as a container or receptacle. As such, it is the "third nature of being" – the first two of which are "the form which is always the same" and "the form which is always in motion":

> This new beginning of our discussion of the universe requires a fuller division than the former, for then we made two classes; now a third must be revealed. The two sufficed for the former discussion. One, which we assumed was a pattern intelligible and always the same, and the second was only an imitation of the pattern, generated and visible. There is also a third kind which we did not distinguish at the time, conceiving that the two would be enough. But now the argument seems to require that we should set forth in words another kind, which is difficult of explanation and dimly seen. What nature are we to attribute to this new kind of being? We reply that it is the receptacle, and in a manner the nurse of all generation [PLAT3: 1176 (48e-49b)].

> And there is a third nature, which is space and is eternal, and admits not of destruction and provides a home for all created things, and is apprehended, when all sense is absent, by a kind of spurious reason, and is hardly real – which we, beholding as in a dream, say of all existence that it must of necessity be in some place and occupy a space, but that what is neither in heaven nor in earth has no existence [PLAT3: 1178-1179 (52a-52b)].

Plato presents us with this idea in his almost-biblical myth of creation, *Timaeus*. Plato's word χώρος – translated here as "space" – carries a connotation of "room" in the sense of "a place (τόπος) for things to be." Elsewhere in *Timaeus* Plato tells us

> But two things cannot be rightly put together without a third; there must be some kind of bond of union between them. And the fairest bond is that which makes the most complete fusion of itself and the things which it combines, and proportion is best adapted to effect such a union. For whenever in any three numbers, whether cube or square, there is a mean, which is to the last term what the first term is to it, and again, when the mean is to the first term as the last term is to the mean – then the mean becoming first and last, and the first and last both becoming means, they will all of them of necessity come to be the same, and having become the same with one another will be all one [PLAT3: 1163 (31b-32a)].

In view of Plato's division of the nature of being into the "world of truth" and the "world of opinion," it is possible to regard Platonic space as the bond or union of these two "natures." And because "proportion is best adapted to effect such a union," Platonic space is tied to, and hardly distinguishable from, the ideas of geometry and geometric means.

> If the universal frame had been created a surface only and having no depth, a single mean would have sufficed to bind together itself and the other terms; but now, as the world must be solid, and solid bodies are always compacted not by one mean but by two, God placed water and air in the mean between fire and earth, and made them to have the same proportion so far as was possible . . . and thus he bound and put together a visible and tangible heaven . . . for this cause and on these grounds he made the world one whole, having every part entire, and being therefore perfect and not liable to old age and disease. And he gave to the world the figure that was most suitable and also most natural . . . Wherefore he made the world in the form of a globe, round as from a lathe, having its extremes in every direction equidistant from the center, the most perfect and most like itself of all figures [PLAT3: 1163-1164 (32a-33b)].

Pragmatically-minded Aristotle seems to have been far less concerned with "space" in this Platonic sense and far more concerned about "place" (τόπος, *topos*). Here we do well to remember that the classical Greeks were first and foremost *realists*. Excepting the Greek atomists, if space and place were to exist at all, they had to "be something." A vacuum or "void" is not something; it is nothing, and both Plato and Aristotle rejected the atomists' idea of the void. For Aristotle, the question of place arises because of locomotion. In his list of the ten categories the word usually translated as "place" is "*pou*" which denotes "where?" The category is not what is meant by "place" (*topos*).

> The physicist must have a knowledge of place, too, as well as of the infinite – namely whether there is such a thing or not, and the manner of its existence and what it is – both because all suppose that which exist are somewhere . . . and because motion in its most general and proper sense is change of place, which we call "locomotion."
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> The question, What is place? presents many difficulties. An examination of all the relevant facts seems to lead to different conclusions. Moreover, we have inherited nothing from previous thinkers, whether in the way of a statement of difficulties or of a solution.
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> The existence of place is held to be obvious from the fact of mutual replacement. Where water now is, there in turn, when the water has gone out as from a vessel, air is present . . . The place is thought to be different from all the bodies which come to be in it and replace one another . . .
>
> Further, the locomotions of the elementary natural bodies – namely, fire, earth, and the like – show not only that place is something, but also that it exerts a certain influence. Each is carried to its own place, if it is not hindered, the one [fire] up, the other [earth] down. Now these are regions or kinds of place – up and down and the rest of the six directions [left, right, before, behind]. Nor do such distinctions (up and down and right and left) hold only in relation to us. To us they are not always the same but change with the direction in which we are turned: that is why the same thing is often both right and left, up and down, before and behind. But in nature each is distinct, taken apart by itself. It is not every chance direction that is up, but where fire and what is light are carried; similarly, too, down is not any chance direction but where what has weight and what is made of earth are carried – the implication being that these places do not differ merely in position, but also as possessing distinct powers . . . Again, the theory that the void exists involves the existence of place; for one would define void as place bereft of body [ARIS6: 354-355 (208a27-208b26)].

We can make some comments at this point regarding the way Aristotle is setting up the problem of "What is place?" First we should note the distinction that place is different from the "body" that occupies it. Although in the passage above Aristotle has not yet confirmed that this is a correct way to view place, that is the conclusion he will make shortly. It is this distinction between place and body-occupying-that-place that produces the serious difficulty in figuring out what a "place" is in a "real" sense. If "place" exists it must be real, in the Greek view, and if it is not a body (i.e. is not composed of the Greek elements), what remains for it to be?

The second interesting point raised above is the idea that place "exerts a certain influence" on bodies. This is a peek into Aristotle's doctrine that bodies have a "natural place" in nature and if not "hindered" will move "into" that natural place. This has been termed Aristotle's "teleology" and is the part of his physics most excoriated by modern scientists. Had Aristotle really been the deist portrayed in the Neo-Platonic and Christian 'Aristotles', this criticism would be justified. But, unlike Plato, he was not. Zeller notes:

> The most important feature of Aristotelian teleology is the fact that it is neither anthropocentric nor is it true to the actions of a creator existing outside the world or even of a mere arranger of the world, but is always thought of as immanent in nature. What Plato effected in the *Timaeus* by the introduction of the world-soul . . . is here explained by the assumption of a teleological activity inherent in nature itself [ZELL: 180].

As we discussed in Chapter 16, modern science has not done away with teleological principles but has merely taken better notice of the rules that must apply to a proper teleological statement of physical principles – namely that any such expression must be capable through mathematical transformation of causal expression in the Margenau sense. Hamilton's principle in integral form is a teleological principle; so too is the second law of thermodynamics; so too is the minimum principle in quantum electrodynamics. All these principles confine themselves to addressing the "How?" question and leave off at the "Why?" question for reasons we have already discussed. Aristotle's teleology is no different, and the flaw in his physics lies in what we would today call its mechanics. The tendency toward "teleological ends" is an hypothesis of a "How?" law "immanent in nature," and here Aristotle and the moderns do not differ in logical essence.

The idea that place "exerts an influence" has another important philosophical implication, namely that "place" is in some way more than merely geometry. This, too, has its modern day counterpart in physics' general theory of relativity (which we will discuss later). In Newtonian physics a body not acted upon by "forces" will continue its motion with uniform velocity in a "straight line." But what is a "straight" line? This has a simple enough definition if the "metric space" used for the mathematical description of "space" is Euclidean, but is a Euclidean metric space a description that accords with what is observed in nature? The finding of the theory of general relativity is that it is not, and that the proper description of the motion of such a body is that it moves along a "geodesic" – which put perhaps too simply could be described as a "physical straight line" (which turns out to be described by curved lines in Euclidean geometry). In the general theory of relativity "matter" determines geodesic lines and "things" (including light) not acted upon by forces travel along geodesic lines. "Gravity" in general relativity is more or less a term that captures the rules of determination of geodesic lines and is neither "force" nor "matter" in the Newtonian sense. It is a "fundamental interaction." Thus, the "curved space" of general relativity is (loosely) said to "exert" or "describe" an "influence" on the motion of things.

Thus far, then, the way Aristotle is setting up the problem is not so far removed from modern theory as is usually supposed. Still, we have so far seen nothing more than the initial set up, much less the solution. Are we justified in presuming that place really exists? Aristotle goes on to say:

> These considerations then would lead us to suppose that place is something distinct from bodies, and that every sensible body is in place . . . If this is its nature, the power of place must be a marvelous thing, and be prior to all other things. For that without which nothing can exist, while it can exist without the others, must needs be first; for place does not pass out of existence when the things in it are annihilated.
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> True, but even if we suppose its existence settled, the question of what it is presents difficulties – whether it is some sort of bulk of body or some entity other than that; for we must first determine its genus.
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> Now it has three dimensions, length, breadth, and depth, the dimensions by which all body is bounded. But the place cannot be body; for if it were there would be two bodies in the same place.

> Further, if body has a place and space, clearly so too have surface and the other limits of body for the same argument will apply to them . . . But when we come to a point we cannot make a distinction between it and its place. Hence if the place of a point is not different from the point, no more will that of any of the others [i.e. the collection of points that define a surface] be different, and place will not be something different from each of them.
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> What in the world, then, are we to suppose place to be? If it has the sort of nature described, it cannot be an element or composed of elements, whether these are corporeal or incorporeal; for while it has size it has not body. But the elements of sensible bodies are bodies, while nothing that has size results from a combination of intelligible elements.
>
> Also we may ask: of what in things is space the cause? None of the four modes of causation can be ascribed to it. It is neither cause in the sense of the matter of existents (for nothing is composed of it), nor as the form and definition of things, nor as end, nor does it move existents.
>
> Further, too, if it is itself an existent, it will be somewhere. Zeno's difficulty demands an explanation; for if everything that exists has a place, place too will have a place, and so on to infinity.
>
> Again, just as every body is in place, so, too, every place has a body in it. What then shall we say about growing things? It follows from these premises that their place must grow with them, if their place is neither less nor greater than they are.
>
> By asking these questions, then, we must raise the whole problem about place – not only as to what it is, but even to whether there is such a thing [ARIS6: 355-366 (208b27-209a30)].

Who of us would have thought that the seemingly obvious idea of "place" should turn out to harbor so many knots in the Cartesian bulrush? Aristotle is pointing out that *how we define* what we mean by "place" has implications for whether such a thing as we define is or is not self-contradictory. Aristotle goes on to slowly dissect the possibilities of what place may be. He finds that it is neither matter nor form because these are not separable from the place-occupying thing, whereas place "itself" is separable from that thing. Rather, place "is supposed to be something like a vessel – the vessel being a transportable place. But the vessel is no part of the thing."

> What then after all is place? The answer to this question may be elucidated as follows.
>
> Let us take for granted about it the various characteristics which are supposed correctly to belong to it. We assume first that place is what contains that of which it is the place, and is no part of the thing; again, that the primary place of a thing is neither less nor greater than the thing; again, that place can be left behind by the thing and is separable; and in addition that all place admits of the distinction of up and down, and each of the bodies is naturally carried to its appropriate place and rests there, and this makes the place either up or down . . .
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> First then we must understand that place would not have been inquired into if there had not been motion with respect to place . . .
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> We say that a thing is in the world, in the sense of place, because it is in the air [for example], and the air is in the world, and when we say it is in the air we do not mean it is in every part of the air, but that it is in the air because of the surface of the air which surrounds it . . .
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> When what surrounds, then, is not separate from the thing, but is in continuity with it, the thing is said to be in what surrounds it, not in the sense of in place but as a part of the whole. But when the thing is separate and in contact, it is primarily in the inner surface of the surrounding body, and this surface is neither a part of what is in it nor yet greater than its extension, but equal to it; for the extremities of things which touch are coincident . . .
>
> It will now be plain from these considerations what place is. There are just four things of which place must be one – the shape, or the matter, or some sort of extension between the extremities, or

> the extremities (if there is no extension over and above the bulk of the body which comes to be in it).
>
> Three of these obviously cannot be . . . Both shape and place, it is true, are boundaries. But not the same thing: the form is the boundary of the thing, the place is the boundary of the body which contains it.
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> The extension between the extremities is thought to be something, because what is contained and separate may often be changed while the container remains the same . . . the assumption being that the extension is something over and above the body displaced. But there is no such extension . . .
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> If there were an extension which were such as to exist independently and be permanent, there would be an infinity of places in the same thing . . . [Aristotle shows that such a definition leads to an infinite regression: the place must have a place, which must have a place, which must & etc.] . . .
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> The matter, too, might seem to be place, at least if we consider it in what is at rest and is not separate but in continuity . . . But the matter, as we said before, is neither separable from the thing nor contains it, whereas place has both characteristics.
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> Well, then, if place is none of the three – neither the form nor the matter nor an extension which is always there, different from, and over and above the extension of the thing which is displaced – place necessarily is the one of the four which is left, namely the boundary of the containing body at which it is in contact with the contained body. (By this contained body is meant what can be moved by way of locomotion).
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> Hence the place of a thing is the innermost motionless boundary of what contains it [ARIS6: 358-361 (210b32-212a20)].

This is a very non-geometrical explanation of "place." The key point with regard to "place" as a boundary is that it is not merely the "innermost boundary of what contains" the thing, but that it is the innermost *motionless* boundary. Place is, as Aristotle goes on to say, "thought to be a kind of surface and, as it were, a vessel, i.e. a container of the thing. Further, place is coincident with the thing, for boundaries are coincident with the bounded." But the *motionless* character of place implies that while the place always contains its movable body, the place itself does not change when the body undergoes locomotion. Place, Aristotle tells us, is "a non-portable vessel." Hence, the place of a boat is not the water in the river in contact with the boat; this is "merely part of a vessel rather than that of place." The place of the boat is instead the entire river "because as a whole it is motionless." We can note that Aristotle did not say that "place" *is* a boundary in an easily-interpreted *geometric* sense; it is "a kind of" surface, which would seem to be a highly abstract generalization of the idea of a "surface."

Such a theory of "place" sounds far more qualitative than quantitative. Aristotelian "place" is not easily reducible to geometric terms, in sharp contrast to Plato's "space." It is rather more like a set theoretic description: the place of the boat is the river; the place of the river is between the banks; the place of the banks . . . etc. Ultimately, the place of the boat is located somewhere in the world (i.e. the universe), but in Aristotle's system this does not present an infinite regression because Aristotle's universe is itself finite. The world "itself" has no "place" because it is ultimately the place with respect to which "places" owe their existence, much like the "reality of a thing" must presuppose an All of Reality as its substratum. The predication of "place" would seem, then, to lead to a system of relationships which in some ways smacks of a "topological" description (i.e., a description in terms of "neighborhoods") that is not altogether incongruent with the very abstract mathematical definition of a "topological space" but falls well short of becoming a "metric space."

This lack of reducibility in terms of a metric space gives Aristotelian "place" a strange and somewhat "non-localizable" character in the sense that the "place of a thing" does not readily admit to description in terms of analytic geometry. Little wonder, then, why Descartes looked upon Aristotelian philosophy with such disfavor! However, lest we rush to conclude that all this is obscure nonsense, it is worthwhile to take note that non-relativistic quantum mechanics has some of this same flavor. "In" an atom the "place" of an electron (so to speak) is an "orbital". Different orbitals are describable in terms of a metric space, and so can be tied to analytic geometry. But the solutions of the Schrödinger equation do not permit an electron to "exist" in the "space between orbitals." (Formally, quantum mechanics says that the probability of finding the electron anywhere except in one of the orbitals is zero). Furthermore, an "orbital" does not actually specify a single "point" in "space" but rather a locus of points, and it is not permitted to "tie" the electron definitely to any one point in this locus at any one moment in time. Finally, an electron can "jump" from one orbital to another, and it is not formally permissible to regard the electron as spending any time in transition wherein it is 'between' orbitals because then it would have to be possible to calculate a non-zero probability of finding it "between" orbitals.[1] I find it difficult to spot how in logical essence such a picture is ontologically less (or more) objectionable than Aristotle's "place" idea, yet I do not regard the quantum theory as flawed by this state of affairs. *Pragmatically* speaking, the modern theory is vastly more fecund and much less vague than what Aristotle was able to achieve in his science, even if ontologically it seems to be no less psychologically "marvelous." Science is pragmatic: If it works, use it; if it doesn't, discard it. I am reminded, though, of the adage about metaphysical glass houses and the throwing of metaphysical stones.

Of a wholly different nature was the theory of the atomists. The founder of atomism was Leucippus, but the main credit for development of the atomist theory goes to his great disciple Democritus (*c*. 460-370 B.C.). Of the ancient Greek atomists, he is the only one who can be favorably compared on anything near an equivalent footing with Aristotle.

[1] The issue of "spending time in transition" is not even a permissible question in quantum mechanics. This is a consequence of Heisenberg's uncertainty principle, which among other things says we cannot make any scientifically valid statements pertaining to the observability of "what happens" during intervals of time shorter than some calculable Δt for some given change in energy ΔE.

Like other Greek thinkers, Democritus held that absolute creation or absolute annihilation was impossible and sought to explain motion (*kinesis*) in the face of this. Because Parmenides had previously "shown" that motion was unthinkable without non-being, Democritus declared that non-being was as good as being. According to Parmenides, "being" was space-filling whereas non-being was "empty." Democritus countered that "the full" and "the empty" were both constituents of all things. It was he who converted non-being into "space." In other words, non-being is not absolute nothing but, rather, is relative non-being. If this seems very familiar to us today, it is because this is by and large the prevalent view today as well. "Space" is the non-being that separates the atoms. For Aristotle the world is a continuous plenum and if "space" means void instead of place, then there is no space, nor is there need for it since *kinesis* is change of form. For Democritus and the other atomists, including Lucretius the Epicurean (97-55 B.C.), atoms are discrete, indivisible plenums and without the void motion is impossible.

> And yet all things are not on all sides jammed together and kept in by body; there is also void in things. To have learned this will be good for you on many accounts . . . If there were not void, things could not move at all; for that which is the property of body, to let and hinder, would be present to all things at all times; nothing could therefore go on, since no other thing would be the first to give way . . . Again however solid things are thought to be, you may yet learn from this that they are of a rare body: in rocks and caverns the moisture of water oozes through and all things weep with abundant drops; food distributes itself through the whole body of living things; trees grow and yield fruit in season, because food is diffused through the whole from the very roots over the stem and all the boughs . . . Once more, why do we see one thing surpass another in weight though not larger in size? For if there is just as much body in a ball of wool as there is in a lump of lead, it is natural it should weigh the same, since the property of body is to weigh all things downward, while on the contrary the nature of void is ever without weight. Therefore when a thing is just as large yet is found to be lighter, it proves sure enough that it has more of void in it; while on the other hand that which is heavier shows that there is in it more of body and that it contains within it much less of void. Therefore that which we are seeking with keen reason exists sure enough, mixed up in things; and we call it void [LUCR: 5].

In the atomists' view, the void is necessary because of the indivisibility and incompressibility of the atoms. A further consequence of this theory is that the world can have no limits, no beginning, and no end. Another consequence is that there are no "forces"; instead the atoms are constantly in a state of rotary motion, coming together, to form atom-complexes, or flying apart. There is no "action at a distance" because the void cannot hinder motion; "hindering" takes place through direct contact from atom to atom. While this view has some facile resemblance to the "particle exchange" paradigm of modern quantum physics, it is thoroughly mechanistic at its roots. The Greek atomists had no "field theory" and relied upon the imputation of a number of fantastic properties attributed to the "atoms."

Atoms came in different "sizes" (though always too small to be seen) and with different weights, weight being regarded as part of the "nature" of the atoms. Democritus' atoms have other intriguing properties as well, including mental and vital properties. Democritus said, "There must be much reason and soul in the air, for otherwise we could not absorb this by breathing."

The atomists' doctrine of space should not be presumed to be the beginning of any geometrical theory of space. The imputed properties of the void are consequences of the properties of the Greek atoms, and there is little evidence that the atomists ever attempted, or thought to attempt, a rigorous geometric treatment of space. This would have been difficult for them in any case. The Greeks possessed Euclid's geometry, but this was not the analytic geometry of today; that invention is credited to Descartes centuries later. All that can be said with confidence of the Greek atomists' void is that it is the real non-being that separates the atoms, whatever that might mean.

§ 1.2 Descartes

As the inventor of analytic (or "coordinate") geometry, we might expect to find Descartes taking the side of the Greek atomists. However, such an expectation would ignore the basic tenets of Descartes' philosophy. In his *Principles of Philosophy* Descartes tells us, "The nature of matter, or of body considered in general, does not consist of being hard, or heavy, or colored . . . but only in the fact that it is a thing possessing extension in length, breadth, and depth . . . The same extension which constitutes the nature of a body constitutes the nature of space . . . not only that which is full of body, but also that which is called a void . . . That a vacuum in the philosophical sense of the word, *i.e.* a space in which there is absolutely no substance, cannot exist is evident from the fact that the extension of space, or internal place, does not differ from the extension of body . . . When we take this word vacuum in its ordinary sense, we do not mean a place or space in which there is absolutely nothing, but only a place in which there are none of those things which we think ought to be in it."

For Descartes body, space, and extension are all one and the same thing. This is a direct consequence of his method, which calls for the denial of reality to any thing that cannot be known either immediately or through an unbreakable series of apodictic deductions.

> Fifthly, we remark that no knowledge is at any time possible of anything beyond those simple natures and what may be called their intermixture or combination with each other. Indeed, it is often easier to be aware of several of them in union with each other than to separate one of them from the others . . .
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> Sixthly, we may say that those natures which we call composite are known by us either because experience shows us what they are, or because we ourselves are responsible for their composition. Matter of experience consists of what we perceive by sense . . . Moreover, we ourselves are responsible for the composition of the things present to our understanding when we believe there is something in them which our mind never experiences when exercising direct perception . . .

> Deduction is thus left to us as the only means of putting things together so as to be sure of their truth. Yet in it, too, there may be many defects. Thus if, in this space which is full of air, there is nothing to be perceived either by sight, touch, or any other sense, we conclude that the space is empty, we are in error, and our synthesis of the nature of a vacuum with that of this space is wrong . . . But it is within our power to avoid this error if, for example, we never interconnect any objects unless we are directly aware that the conjunction of the one with the other is wholly necessary. Thus we are justified if we deduce that nothing can have figure which has not extension, from the fact that figure and extension are necessarily conjoined [DESC2: 22-23].

Space in and of itself, as an abstraction, has no physical reality in Descartes' view. In his example above, he points out that the space before us is "full of air" and our mere inability to perceive the air is no ground for concluding that what is before us is something to be called empty space. We might protest that no one would make such an error because, for example, we can feel the wind on our bodies (and that therefore the example is contrived). However, this would be a false argument because the object under consideration is not in the same place as is the body that feels the wind. We would not directly know if there was an absence of air or other matter in a place other than where we presently stand. A modern day disciple of Descartes would be able to use this line of reasoning to argue against the idea that "outer space" beyond the earth's atmosphere can be known to "really" be a vacuum. We could not be sure there were not "bodies" (substances) undetectable by our senses yet possessing "extension" (the only true "nature" of a "substance" according to Descartes) and "filling" that which we call space. Therefore we cannot know by any means that in reality such a thing as a vacuum exists, therefore we must deny there is a vacuum.

This is not as far-fetched as it might sound. Descartes of course knew nothing about our modern ideas of gravitational or electromagnetic fields, but had he known of these it would not have been hard for him to argue (were he convinced by perceptible evidence that they existed) that they were extended substances. Indeed, it is hard to visualize these ghostly objects of scientific theory in any way other than this. Descartes was, in fact, a pioneer in the science of optics, and although we could argue that darkness is the absence of light, the absence of one "substance" in perception does not imply emptiness of all "substances." Thus "darkness" does not imply "nothingness."

> By extension we understand whatever has length, breadth, and depth, not inquiring whether it be a real body or merely space; nor does it appear to require further explanation, since there is nothing more easily perceived by our imagination. Yet the learned frequently employ distinctions so subtle that the light of nature is dissipated in attending to them, and even those matters of which no peasant is ever in doubt become invested in obscurity. Hence we announce that by extension we do not here mean anything distinct and separate from the extended object itself; and we make it a rule not to recognize those metaphysical entities which really could not be presented to the imagination. For even though someone could persuade himself, for example, that supposing every extended object in the universe were annihilated, that would not prevent extension in itself alone existing, this conception of his would not involve the use of any corporeal image, but would be based on a false judgment of the intellect working by itself. He will admit this himself, if he reflect attentively on

> this very image of extension when, as will then happen, he tries to construct it in his imagination. For he will notice that, as he perceives it, it is not divested of a reference to every object, but that his imagination of it is quite different from his judgment about it [DESC2: 29].

Descartes picks apart three viewpoints of extension: 1) extension occupies place; 2) body possesses extension; and 3) extension is not body. Of the first he writes that "extension" may be substituted "for that which is extended" and, "My conception is entirely the same if I say *extension occupies place*, as when I say *that which is extended occupies place* . . . This explains why we announced that here we would treat of extension, preferring that to 'the extended,' although we believe that there is no difference in the conception of the two."

Of the second statement he writes

> Here the meaning of *extension* is not identical with that of body, yet we do not construct two distinct ideas in our imagination, one of body, the other of extension, but merely a single image of extended body; and from the point of view of the thing it is exactly as if I had said: *body is extended*, or better, *the extended is extended*. This is a peculiarity of those entities which have their being merely in something else, and can never be conceived without the subject in which they exist. How different it is with those matters which are really distinct from the subjects of which they are predicated. If, for example, I say *Peter has wealth*, my idea of Peter is quite different from that of wealth . . . Failure to distinguish the diversity between these two cases is the cause of error of those numerous people who believe that extension contains something distinct from that which is extended [DESC2: 30].

As for the third statement, Descartes points out that extension distinct from body cannot be "grasped by the imagination." Consequently "extension" so viewed can only be conceived

> by means of a genuine image. Now such an idea necessarily involves the concept of body, and if they say that extension so conceived is not body, their heedlessness involves them in the contradiction of saying that *the same thing is at the same time body and not body* [DESC2: 30].

We see from all this that Descartes fixes extension, which he takes to be the same thing as space, to the "body" which is extended. Thus Descartes not only disagrees with the Greek atomists but also with both Aristotle and Plato as to the "nature" of space. Aristotle argued that if place was a property of the thing (body) that would make place either part of the matter of the body or of its form. In his *Physics* he mounted arguments that place could be neither of these. Descartes declines to enter in to any such matter vs. form subtleties. For him the property that declares the existence of a "body" is its "easily perceived" extension and no more need be said of it. All further disputation he holds to be a matter of words and definitions.

> We ought not to judge so ill of our great thinkers as to imagine that they conceive the objects themselves wrongly, in cases where they do not employ fit words in explaining them. Thus when some people call *place* the *surface of the surrounding body*, there is no real error in their conception; they merely employ wrongly the word *place*, which by common use signifies that

> simple and self-evident nature in virtue of which a thing is said to be here or there. This consists wholly in a certain relation of the thing said to be in the place towards the parts of space external to it [DESC2: 26].

What, then, of geometry? We have seen that Plato's space is nearly indistinguishable from geometry, a philosophical position that bespeaks of the Pythagorean character of his philosophy. Aristotle's "place" can barely be tied to geometry in any quantitative sense at all. What both men share in common is the persistent realism of the classical Greeks. Descartes' position on this issue is more properly called idealism. When we are thinking in terms of geometrical or mathematical abstractions, he tells us, it is vital in the prevention of error to always bear in mind that ultimately these abstractions have no meaning except insofar as they ultimately must keep a reference to the "bodies" that they *describe*. But if we keep this reference in mind, these mathematical abstractions *used as reductions* are an aid to our understanding. Descartes does not divorce geometry from physics, which geometry serves as an aid to understanding. But it is a fundamental error to attribute physical reality to geometric or arithmetic quantities independently of bodies.

> It is likewise of great moment to distinguish the meaning of the enunciations in which such names as *extension, figure, number, superficies, line, point, unity*, etc. are used in so restricted a way as to exclude matters from which they are not really distinct . . .
>
> But we should carefully note that in all other propositions in which these terms, though retaining the same signification and employed in abstraction from their subject matter, do not exclude or deny anything from which they are not really distinct, it is both possible and necessary to use the imagination as an aid. The reason is that even though the understanding in the strict sense attends merely to what is signified by the name, the imagination nonetheless ought to fashion a correct image of the object, in order that the very understanding itself may be able to fix upon other features belonging to it that are not expressed by the name in question, whenever there is occasion to do so, and may never impudently believe that they have been excluded . . . Does not your Geometrician obscure the clearness of his subject by employing irreconcilable principles? He tells you that lines have no breadth, surfaces no depth; yet he subsequently wishes to generate the one out of the other, not noticing that a line, the movement of which is conceived to create a surface, is really a body; or that, on the other hand, the line which has no breadth is merely a mode of body.
>
> Recognition of this fact throws much light on Geometry, since in that science almost everyone goes wrong in conceiving that quantity has three species, the line, the superficies, and the solid. But we have already stated that the line and the superficies are not conceived as being really distinct from solid bodies or from one another. Moreover, if they are taken in their bare essence as abstractions of the understanding, they are no more diverse species of quantity than the "animal" and "living creature" in man are diverse species of substance [DESC2: 30-31].

Descartes championed the method of understanding phenomena by means of mathematical descriptions, and he was among the first to give physics its modern mathematical footing. However, Descartes was no Platonist, a point that today is sometimes overlooked. Geometry he views as an aid to understanding nature, but space is not geometry and physics can never be made subordinate to geometry (or mathematics generally) without introducing errors in understanding.

§ 1.3 Newton and Leibniz

In describing Sir Isaac Newton, John Maynard Keynes wrote

> In the eighteenth century and since, Newton came to be thought of as the first and greatest of the modern age of scientists, a rationalist, one who taught us to think on the lines of cold and untinctured reason. I do not see him in this light. I do not think that anyone who has pored over the contents of that box which he packed up when he finally left Cambridge in 1696 and which, though partially dispersed, have come down to us, can see him like that. Newton was not the first of the age of reason. He was the last of the magicians, the last of the Babylonians and Sumerians, the last great mind which looked out on the visible and intellectual world with the same eyes as those who began to build our intellectual inheritance rather less than 10,000 years ago. Isaac Newton, a posthumous child born with no father on Christmas Day, 1642, was the last wonderchild to whom the Magi could do sincere and appropriate homage.[1]

The box to which Keynes refers is the collection of Newton's secret papers known today as "The Portsmouth Papers." No one denies that Newton is one of the greatest scientists who has ever lived; but as a philosopher Newton was a mystic. The most basic underpinnings of his theory are founded upon a mystic's view of God and not, as some suppose, upon the British empiricism that was born during Newton's own lifetime. Nor was Newton the first positivist, although both empiricism and positivism have, at various times, held him up as the very exemplar of positions adopted by these schools of thought. Nowhere are the mystical roots of Newton's thought better demonstrated than by his conception of space.

In his *Principia* Newton writes

> Hitherto I have laid down the definitions of such words as are less known, and explained the sense in which I would have them to be understood in the following discourse. I do not define time, space, place, and motion, as being well known to all. Only I must observe, that the common people conceive those quantities under no other notions but from the relations they bear to sensible objects. And thence arise certain prejudices, for the removing of which it will be convenient to distinguish them into absolute and relative, true and apparent, mathematical and common.
>
> II. Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is commonly taken for immovable space; such is the dimension of a subterraneous, an aerial, or celestial space determined by its position in respect of the earth. Absolute and relative space are the same in figure and magnitude; but they do not remain always numerically the same . . .
>
> III. Place is a part of space which a body takes up, and is according to the space either absolute or relative. I say, a part of space; not the situation, nor the external surface of a body . . .
>
> IV. Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another . . .
>
> As the order of time is immutable, so also is the order of the parts of space. Suppose those parts to be moved out of their places, and they will be moved (if the expression may be allowed) out of

[1] Jeremy Bernstein, *Einstein*, NY: The Viking Press, 1973, pg. 167.

> themselves. For times and spaces are, as it were, the places as well of themselves as of all other things. All things are placed . . . in space as to order of situation. It is from their essence or nature that they are places; and that the primary places of things should be movable is absurd. These are therefore the absolute places; and translation out of those places are the only absolute motions.[2]

It is no doubt clear Newton's assertion that the definition of space is "well known to all" is untrue. In Newton's theory absolute space is held to exist as a thing and independently of all other things. Relative space, the space described in geometry, is merely "some movable dimension or measure" of absolute space.

> But because the parts of space cannot be seen or distinguished from one another by our senses, therefore in their stead we use sensible measures of them. For from their positions and distances from any body considered as immovable we define all places; and then with respect to such places we estimate all motions, considering bodies as transferred from some of those places into others. And so, instead of absolute places and motions, we use relative ones, and that without inconvenience in common affairs; but in philosophical disquisitions we ought to abstract from our senses and consider things themselves, distinct from what are only sensible measures of them.[3]

Absolute space, for Newton, functions as a kind of real substratum for relative space. In one way this is understandable in the same way as the idea of the reality of a thing must presuppose an All of Reality within which reality of a thing is a limitation. Note, however, that Newton *reifies* his absolute space; it is a *thing* existing physically and independently of other physical things.

And what is the "nature" of this absolute space that our senses can in no way detect?

> This most beautiful system of the sun, planets, and comets could only proceed from the counsel and dominion of an intelligent and powerful Being. And if the fixed stars are the centers of other like systems, these . . . must all be subject to the dominion of the One . . .
>
> This Being governs all things, not as the soul of the world, but as Lord over all; and on account of his dominion He is wont to be called *Lord God* . . . He endures forever and is everywhere present; and by existing always and everywhere, he constitutes duration and space . . . He is omnipresent not *virtually* only, but also *substantially* . . . In Him are all things contained and moved; yet neither affects the other: God suffers nothing from the motion of bodies; bodies find no resistance from the omnipresence of God. It is allowed by all that the supreme God exists necessarily; and by the same necessity He exists *always* and *everywhere* . . . He is utterly void of all body and bodily figure, and can therefore neither be seen, nor heard, nor touched; nor ought He to be worshiped under the representation of any corporeal thing. We have ideas of His attributes, but what the real substance of anything is we know not . . . We know Him only by his most wise and excellent contrivances of things, and final causes . . . All that diversity of natural things which we find suited to different times and places could arise from nothing but the ideas and will of a Being necessarily existing.[4]

Newton stops just short of identifying space (and also his absolute Time) with God. However, the distinction is difficult to draw. As a spokesman for Newton, the Reverend Samuel Clarke, wrote to Leibniz, ". . . space and duration are not *hors de Dieu*, but are caused by, and are immediate

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[2] Isaac Newton, *Mathematical Principles of Natural Philosophy*, Definitions.
[3] *ibid*.
[4] Isaac Newton, *op cit.*, Bk. III, General Scholium.

and necessary consequences of His existence".[5] Here we clearly see Newton's transcendent retreat into mysticism.

Leibniz' view of space is, to put it mildly, as opposite to Newton's as one is likely to find. For Leibniz space is not a thing in any "substantial" sense; rather, it is merely an ordering relation and, furthermore, a merely *relative* ordering relation between real things. Thus space is an ideal representation and the only proper context for it is mathematical. If the spatial relation between two things was itself a thing then space would have to have the peculiar property of being a substance with, so to speak, its left foot in one thing and its right foot in another. Leibniz regards this as absurd. Furthermore, because they are ideal and mathematical representations, all spatial relations must be relative to the things they relate and there is no absolute space nor absolute motion in the Newtonian sense.

Leibniz' point of departure from the views of his contemporaries stems from the issue of whether or not "extension" is the fundamental property of a body. We have seen that Descartes took the position that it was. "Corpuscularians" like Newton also held to this view. Leibniz dissents.

> First we would have to be sure that bodies are substances and not just true phenomena, like the rainbow. But assuming they are, I think we can show that a corporeal substance does not consist in extension or in divisibility. For you will grant me that two bodies which are at a distance – for example two triangles – are not really one substance. But now let us suppose that they come together to form a square: can merely being in contact make them into a substance? I don't think so. But every extended mass can be considered as made up of two, or a thousand, others. Extension comes only from contact. Thus you will never find a body of which we can say that it is truly a substance; it will always be an aggregation of many substances. Or rather, it will never be a real being, since the parts which make it up face just the same difficulty, and so we never arrive at a real being, because beings by aggregation can have only as much reality as their ingredients. From this it follows that the substance of a body – if they have them – must be indivisible, and it doesn't matter whether we call that a soul or a form . . . Extension is an attribute which could never make up a complete being; and we could never get from it any action or change; it only expresses a present state, and never the future or the past, as the notion of a substance must do [LEIB9: 115-116].

What does this have to do with space? Atomists from Democritus to Newton argued that space (the void, the vacuum) must exist or else motion would be impossible. Corpuscles are characterized by their extension, and hence must also be hard (incompressible) bodies. Without the vacuum they would be packed together like sardines in a can and prevent each other from moving. Take away extension as the fundamental property of a body, however, and the vacuum is unnecessary.

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[5] Jeremy Bernstein, *op cit.*, pg. 83.

> If the world were full of hard particles which could be neither bent nor divided, as atoms are represented, then motion would indeed be impossible[6]. But in fact hardness is not fundamental; on the contrary fluidity is the fundamental condition, and the division into bodies is carried out – there being no obstacle to it – according to our need. That takes all the force away from the argument that there must be a vacuum because there is motion [LEIB1a: 151].

We have seen earlier that, for Leibniz, to be a substance is to be a monad, an incorporeal entity. Leibniz argues that "extension" is an abstraction, not a "real thing" at all. Thus "fluidity" rather than extension is "the fundamental condition" as quoted above. It is the abstract "nature" of "extension" that makes space merely a relationship.

> 'Place' is either *particular*, as considered in relation to this or that body, or *universal*; the latter is related to everything, and in terms of it all changes of every body whatsoever are taken into account. If there were nothing fixed in the universe, the place of each thing would still be determined by reasoning, if there were a means of keeping a record of all the changes or if the memory of a created being were adequate to retain them . . . Extension is an abstraction from the extended, and the extended is a continuum whose parts are coexistent, i.e. exist at the same time . . . Some people have thought that God is the place of objects . . . but it makes place involve something over and above what we attribute to space, to which we deny agency. Thus viewed, space is no more a substance than time is, and if it has parts it cannot be God. It is a relationship: an order, not only among existents, but also among possibles as though they existed [LEIB1a: 149].

Leibniz' remark about "possibles" bears further scrutiny. Let us remind ourselves that Leibniz is a rationalist; sense impression alone does not constitute our knowledge of things. The mind is no "wax tablet" or *tabula rasa*. Our understanding of things involves mental additives (rational ideas) and these must be taken into account. So far as "real bodies" are concerned,

> Body could have its own extension without implying that the extension was always determinate or equal to the same space. Still, although it is true that in conceiving body one conceives something in addition to space, it does not follow that there are two extensions, that of space and that of body. Similarly, in conceiving several things at once one conceives something in addition to the number, namely the things numbered; and yet there are not two pluralities, one of them abstract (for the number) and the other concrete (for the things numbered). In the same way, there is no need to postulate two extensions, one abstract (for space) and the other concrete (for body). For the concrete one is as it is only by virtue of the abstract one: just as bodies pass from one position in space to another, i.e. change how they are ordered in relation to one another, so things pass also from one position to another within an ordering or an enumeration . . . In fact, time and place are only kinds of order; and an empty place within one of these orders (called 'vacuum' in the case of space), if it occurred, would indicate the mere possibility of the missing item and how it relates to the actual [LEIB1a: 127].

The "abstract conception" of space determines the "concrete conception" of the body, and does so in accordance with the law of continuity. In like fashion "empty space" is the conception of a relationship in which is contained "the mere possibility of the missing item." The ability to conceive of emptiness does not implicate the existence of that emptiness as a real thing. That would be the same as saying, "nothing is something, nothing is a thing."

[6] if there were no vacuum.

> I hold that time, extension, motion, and in general all forms of continuity as dealt with in mathematics are only ideal things; that is to say that, just like numbers, they express possibilities . . . But to speak more accurately, extension is the order of *possible coexistences*, just as time is the order of *inconsistent* but nevertheless connected *possibilities*, such that these orders relate not only to what is actual, but also to what could be put in its place, just as numbers are indifferent to whatever may be counted. Yet in nature there are no perfectly uniform changes such as are required by the idea of movement which mathematics gives us, any more than there are actual shapes which exactly correspond to those which geometry tells us about. Nevertheless, the actual phenomena of nature are ordered, and must be so, in such a way that nothing ever happens in which the law of continuity . . . or any of the other most exact mathematical rules, is ever broken. Far from it: for things could only ever be made intelligible by these rules, which alone are capable . . . of giving us insight into the reasons and intentions of the author of things [LEIB10: 252-253].

The Platonic flavor of Leibniz' theory is clearly in evidence in these arguments. Space (and time) are abstractions made known, in the rationalists' view, by innate ideas. For Leibniz and the other rationalists of his time, mathematical ideas are among the store of objective knowledge *a priori* by which we can have "insight" into things. It is a short step from here to the view that the only proper and possible explanation for space is geometrical.

> This *tabula rasa* of which one hears so much is a fiction, in my view, which nature does not allow and which arises solely from the incomplete notions of philosophers – such as vacuum, atoms, the state of rest (whether absolute, or of two parts of a whole relative to each other), or such as that prime matter which is conceived without any form. Things which are uniform, containing no variety, are always mere abstractions: for instance, time, space, and the other entities of pure mathematics. There is no body whose parts are at rest, and no substance which does not have something which distinguishes it from every other . . . And I think I can demonstrate that every substantial thing, be it soul or body, has a unique relationship to each other thing [LEIB1a: 109-110].

Thus, for Leibniz, there is no distinction between space and the pure mathematics of geometry. We can trust in these "abstractions" because the innate ideas of geometry – meaning in Leibniz' day Euclidean geometry – are self-evident truths in the axioms and true rational deductions in the theorems. From the drafting of Leibniz' *New Essay* it would be another 150 years before a mathematician named Riemann kicked over this rationalist applecart.

§ 1.4 Maxwell and Einstein

Leibniz wrote his *New Essay* with the intent of engaging Locke in a philosophical debate. Locke's death in 1704 aborted this plan, and Leibniz subsequently left the *New Essay* unpublished, feeling that it was unfair to publish a criticism of the views of a man who could no longer defend them. (R.E. Raspe published the *New Essay* in 1765, 49 years after Leibniz' death). In the years that followed, the overwhelming success of Newton's physics remade the scientific world. By the nineteenth century, when positivism held sway over science and supernatural explanations were no longer acceptable, Newton's ideas of absolute space and absolute time were firmly entrenched in the world of physics – his mystic and theological underpinnings of these ideas simply ignored by the physics community.

Now, Newton had held that absolute space was beyond our perceptual ability, but by the latter half of the nineteenth century the new theory of electromagnetism seemed to make it possible to carry out a direct experimental verification of the existence of absolute space. For although absolute space "itself" was a vacuum, this was not taken to mean that space was *empty* (not filled). Newton, Huygens, and all subsequent physicists until Einstein held that space had to be filled with a strange substance called "the æthereal medium" or, more briefly, "the æther," which was presumed to exist in a state of absolute rest with respect to absolute space.

> QU. 18. . . . Is not the heat of the warm room conveyed through the vacuum by the vibrations of a much subtler medium than air, which after the air was drawn out remained in the vacuum? And is not this medium the same with that medium by which light is refracted and reflected, and by whose vibrations light communicates heat to bodies, and is put into fits of easy reflexion and easy transmission? And do not the vibrations of this medium in hot bodies contribute to the intenseness and duration of their heat? And do not hot bodies communicate their heat to contiguous cold ones by the vibrations of this medium propagated from them into the cold ones? And is not this medium exceedingly more rare and subtle than the air, and exceedingly more elastic and active? And doth it not readily pervade all bodies? And is it not (by its elastic force) expanded through all the heavens?
>
> QU. 19. Doth not the refraction of light proceed from the different density of this æthereal medium in different places, the light receding always from the denser parts of the medium? And is not the density thereof greater in free and open spaces void of air and other grosser bodies, than within the pores of water, glass, crystal, gems, and other compact bodies? . . .
>
> QU. 20. Doth not this æthereal medium in passing out of water, glass, crystal, and other compact and dense bodies into empty spaces grow denser and denser by degrees, and by that means refract the rays of light not in a point but by bending them gradually in curved lines? And doth not the gradual condensation of this medium extend to some distances from the bodies, and thereby cause the inflexions of the rays of light, which pass by the edges of dense bodies, at some distance from the bodies?
>
> QU. 21. Is not this medium much rarer within the dense bodies of the Sun, stars, planets and comets than in the empty celestial spaces between them? And in passing from them to great distances, doth it not grow denser and denser perpetually, and thereby cause the gravity of those great bodies towards one another, and of their parts towards the bodies, every body endeavoring to go from the denser parts of the medium towards the rarer? . . .
>
> QU. 22. May not planets and comets, and all gross bodies, perform their motions more freely, and with less resistance in this æthereal medium than in any fluid, which fills all space adequately without leaving any pores, and by consequence is much denser than quick-silver or gold? And may not its resistance be so small as to be inconsiderable? . . .
>
> QU. 24. Is not animal motion performed by the vibrations of this medium, excited in the brain by the power of will, and propagated from thence through the solid, pellucid and uniform *capillamenta* of the nerves into the muscles for contracting and dilating them?[7]

To be fair to Newton, he never said that he had any proof of the existence of the æther. But it is clear enough that the æther was important if Newton's mechanistic theory of physics was to have

[7] Isaac Newton, *Optics*, BK III, Pt. 1, "Queries."

much of any chance to establish a mechanistic basis for his corpuscular theory of light, and for avoiding the menace of action-at-a-distance for the phenomenon of gravity. In a letter to Bentley, later quoted by Faraday, Newton wrote,

> That gravity should be innate, inherent, and essential to matter, so that one body may act upon another at a distance through a *vacuum* without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity that I believe no man who has in philosophical matters a competent faculty of thinking can ever fall into it.

Huygens, too, believed the æther had to exist as the seat of vibrations for his wave theory of light. So, too, did every noteworthy theoretical physicist, right up to and including Maxwell and beyond. If space was indeed a "real thing" whose dominant property was that of being an emptiness, then some material "medium" had to fill it in order to explain mechanistically the behavior of light, gravity, heat, and a number of other phenomena. Nineteenth century physicists were, naturally, quite reticent to claim any true understanding of "the æther" mechanism; that would not have been good "positive science." But, as Maxwell noted, the æther "can not be gotten rid of." Maxwell developed his theory by analogy to a mental model of mechanistic behaviors that Feynman called, "a model of idler wheels and gears and so on in space." Maxwell himself wrote,

> I think we have good evidence for the opinion that some phenomenon of rotation is going on in the magnetic field, that this rotation is performed by a great number of very small portions of matter, each rotating on its own axis, this axis being parallel to the direction of the magnetic force, and that the rotations of these different vortices are made to depend upon one another by means of some kind of mechanism connecting them.
>
> The attempt which I then made to imagine a working model of this mechanism must be taken for no more than it really is, a demonstration that a mechanism may be imagined capable of producing a connexion mechanically equivalent to the actual connexion of the parts of the electromagnetic field. The problem of determining the mechanism required . . . always admits to an infinite number of solutions.[8]

One of the most glorious outcomes of Maxwell's theory was that it explained light as being simply a propagating electromagnetic wave. This result toppled Newton's corpuscular model of light and appeared to give stunning theoretical confirmation to Huygens' theory. The theory predicted the radiation of electromagnetic waves, and this prediction was confirmed experimentally several years later by Heinrich Hertz, leading to the invention of the antenna and, ultimately, of radio. The theory also gave a precise quantitative value for the velocity at which light propagates through empty space (roughly 186,000 miles per second), and this prediction, too, was later confirmed by experiment.

[8] James Clerk Maxwell, *A Treatise on Electricity and Magnetism*, vol. 2, Ch. XXI, art. 831.

But the theory also seemed to provide for the possibility of actually verifying the existence of Newton's absolute space. The equations providing the value for the velocity of light come out in a very special mathematical form. They are, to use technical language, "invariant to coordinate transformation." This means that the predicted value of the velocity of light did not depend in any way on any reference to any kind of "inertial" relative motion. Thus it was concluded that this velocity could only be an *absolute* velocity, i.e. a velocity with respect to absolute space. Furthermore, since this velocity was taken to be the velocity at which light propagated through the æther, this further implied that the æther itself must be at rest with respect to absolute space. Because the earth is *not* at rest with respect to absolute space[9], this meant it would be possible to measure the earth's velocity with respect to absolute space, thereby *experimentally* confirming the existence of absolute space.

The means for making this measurement was invented by American physicist Albert Michelson, who published the results of his first attempt in 1881. The first Michelson experiment, to the surprise of everyone (or, at least, everyone in physics), gave a null result. It was soon found that an oversight had been made in Michelson's calculations which could have accounted for the finding of the null result. Thereupon Michelson made a correction to his apparatus and repeated the experiment in collaboration with American chemist Edward Morley. The result was published in 1887, and again the expected effect of the earth's motion relative to absolute space failed to appear. This time there was no mistake, and the Michelson-Morley experiment stunned the world of physics to its foundations.

Physicists made a number of bold conjectures attempting to explain the null result of the Michelson-Morley experiment. Michelson himself proposed that perhaps the æther was not at rest with respect to absolute space at all; perhaps the motion of the earth "drags" the æther along with it, at least in the vicinity of the earth. But this hypothesis has other consequences that turned out to not happen when physicists tested for them. Other hypotheses for explaining the null result were put forward as well. One possibility was that the motion of bodies relative to the æther caused changes in the shape of electrons and atoms such that materials contracted in the direction of motion with respect to the æther. This hypothesis was favored by the Dutch physicist H.A. Lorentz, who was one of the leading physicists of the time, and it came to be known as the Lorentz force. However, this theory was not free of difficulties and objections, which Lorentz himself was willing to openly acknowledge even as he sought a way to answer them:

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[9] It is mathematically impossible for the earth to be at rest with respect to absolute space because the velocity of any place on the face of the earth is not constant; it changes direction as the earth revolves about its axis and orbits around the sun.

> The experiments of which I have spoken are not the only reason for which a new examination of the problems connected with the motion of the Earth is desirable. Poincaré has objected to the existing theory of electric and optical phenomena in moving bodies that, in order to explain Michelson's negative result, the introduction of a new hypothesis has been required, and that the same necessity may occur each time new facts will be brought to light. Surely this course of inventing new hypotheses for each new experimental result is somewhat artificial. It would be more satisfactory if it were possible to show by means of certain fundamental assumptions and without neglecting terms of one order of magnitude or another, that many electromagnetic actions are entirely independent of the motion of the system. Some years ago, I already sought to frame a theory of this kind.[10]

It is always possible, by means of *ad hoc* tinkering with hypotheses, to produce equations that "curve fit" a theory to match experimental findings. The present day Big Bang theory of the cosmos does this almost every time a new finding comes to light. Such tinkering is never purely mathematical because mathematics follows rules that stem from its basic axioms, and connecting mathematics to the world of physics takes place only through the decisions made by physicists regarding what factors are to be mathematically expressed. The introduction of new hypotheses usually involves guesses pertaining to the ontological matters with which physics deals. Unrestrained inventing of new hypotheses always threatens to turn physics into a hotchpotch aggregate of special case rules. The one tangible benefit to the era of positivism in science was that "positive science" demanded that all such *ad hoc* hypotheses be treated as "guilty until proven innocent." Nineteenth century science did not report mere speculations in the newspapers or parade them to lay public, like a politician garnering votes, to get them elected to office.

Einstein's solution of the æther problem involved not so much the introduction of a new hypothesis as it did the rejection of an old one, namely Newton's hypotheses of absolute space and absolute time. As mentioned earlier, Maxwell's equations are not tied to any special geometrical "frame of reference" and keep their mathematical form regardless of whatever the state of relative motion of an observer of electromagnetic phenomena may be, provided only that this state of motion is unaccelerated. The same is not true of Newtonian mechanics, where the mathematical forms of the equations of mechanics do depend on the velocity of the observer. (This is one mathematical reason for the hypothesis of Newtonian absolute space and absolute time). The source of this variation is the old law of "velocity addition" credited to Galileo. It was this variability of the mathematical form of the laws of physics that Einstein challenged.

> Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics

[10] H.A. Lorentz, "Electromagnetic phenomena in a system moving with any velocity less than that of light," *Proceedings of the Academy of Sciences of Amsterdam*, vol. 6, 1904.

> and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity *c* which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell's theory for stationary bodies. The introduction of a "luminiferous æther" will prove to be superfluous inasmuch as the view here to be developed will not require an "absolutely stationary space" provided with special properties, nor assign a velocity-vector to a point of the empty space in which electromagnetic processes take place.[11]

Einstein's first postulate basically amounts to saying that the laws of physics cannot be made to depend on the coordinate system of the geometry in which they are expressed. Instead, they must hold good for all observers regardless of whether an observer is regarded as not himself moving or is regarded as moving at some uniform velocity. In other words, Einstein is doing away with the need for Newton's absolute space. This is not a new rule regarding *ontological* matters of physics; it is instead a "law about laws." It is a rule that constrains the mathematical forms by which the equations of physics are to be expressed. Einstein's second postulate is not so much a new postulate regarding "matter" as it is a rejection of an earlier postulate, namely the one which held that the velocity of light as predicted from Maxwell's theory was to be interpreted as an absolute velocity with respect to absolute space. Einstein's position in effect was equivalent to interpreting the Maxwell result as predicting that every observer would always observe exactly the same velocity of light regardless of whatever his inertial reference frame might be. This second postulate clashed with the Galilean law of velocity addition, and Einstein's problem, therefore, was to explain where Galileo's law went wrong.

Einstein pointed out that such elementary spatial ideas as position, length, and velocity really have no physical meaning except in terms of the procedures and processes by which they are actually measured. Similarly, the idea of "time" likewise has no meaning for a physicist except in terms of the procedures and processes by which "time" is measured. He therefore undertook a piercing examination of these processes. Now any such process is fundamentally a process of coordination by which one observable entity or event (e.g. the motion of a body) is related to another entity or event (e.g. the reading of a measuring instrument such as a yardstick or a clock). Einstein undertook to establish the rules of correspondence that must govern measurement processes of this sort if the Principle of Relativity is to be satisfied. The result was nothing less than a previously unexpected set of requirements by which the mathematical ideas of *geometry* (not "space") must be employed in physics. Among other things, this led to the development of a

[11] A. Einstein, "On the electrodynamics of moving bodies," *Annalen der Physik*, 17, 1905, in *The Principle of Relativity*, W. Perrett & G.B. Jeffery (tr.), NY: Dover Publications, 1952.

new definition of something called “space-time,” which the *Oxford Dictionary of Physics* defines as “a geometry that includes the three dimensions and a fourth dimension of time.”

Einstein’s 1905 paper, quoted above, was restricted to the special case of systems that were not undergoing acceleration of any kind. This theory is known as the “special” theory of relativity because of this restriction. Obviously this “special” theory had to be extended to include the effects of acceleration; it took Einstein another 11 years to complete this extension. Why did this take so long? As it happens, a host of thorny issues and problems, most dealing with very fundamental ideas that most of us normally take for granted, accompanied this extension. The special theory of relativity did away with the idea of absolute velocity, but its reach did not extend to the idea of *absolute* acceleration. And, of course, the idea of absolute acceleration is meaningless except in relation to Newton’s absolute space. In classical physics “forces” produce acceleration, and if there is no absolute space then the laws of space-time geometry that describe accelerations must not depend on special “privileged” reference frames that rely, overtly or covertly, on the idea of an absolute space.

> In a homogeneous gravitational field (acceleration of gravity γ) let there be a stationary system of co-ordinates K, oriented so that the lines of force of the gravitational field run in the negative direction of the axis of *z*. In a space free of gravitational fields let there be a second system of co-ordinates K’, moving with uniform acceleration (γ) in the positive direction of its axis *z* . . . Relatively to K, as well as relatively to K’, material points which are not subjected to the action of other material points move in keeping with the equations
>
> $$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 0, \quad \frac{d^2z}{dt^2} = -\gamma \; .$$
>
> For the accelerated system K’ this follows directly from Galileo’s principle, but for the system K, at rest in a homogeneous gravitational field, from the experience that all bodies in such a field are equally and uniformly accelerated. This experience . . . is one of the most universal which the observation of nature has yielded; but in spite of that the law has not found any place in the foundations of our edifice of the physical universe.
>
> But we arrive at a very satisfactory interpretation of this law of experience if we assume that the systems K and K’ are physically exactly equivalent, that is, if we assume that we may just as well regard the system K as being in a space free from gravitational fields, if we then regard K as uniformly accelerated. This assumption of exact physical equivalence makes it impossible for us to speak of the absolute acceleration of the system of reference, just as the usual theory of relativity forbids us to talk of the absolute velocity of a system, and it makes the equal falling of all bodies in a gravitational field seem a matter of course.[12]

In Newton’s theory the acceleration due to gravity occupied precisely a privileged frame of reference; Newton, after all, did not introduce absolute space from whim. Newtonian gravitation produces an absolute acceleration; this is what Einstein meant when he said this law “has not found any place in the foundations of our edifice of the physical universe.” Einstein’s 1911

[12] A. Einstein, “On the influence of gravitation on the propagation of light,” *Annalen der Physik*, 35, 1911.

attempt to solve the problem was not successful, as he himself realized. It took him another five years to find his way through the difficulties that a general theory of relativity must overcome. His breakthrough came at an unexpected place.

> The special theory of relativity thus does not depart from classical mechanics through the postulate of relativity, but through the postulate of the constancy of the velocity of light *in vacuo*, from which, in combination with the special principle of relativity, there follow, in the well-known way, the relativity of simultaneity, the Lorentzian transformation, and the related laws for the behavior of moving bodies and clocks.
>
> The modification to which the special theory of relativity has subjected the theory of space and time is indeed far-reaching, but one important point has remained unaffected. For the laws of geometry, even according to the special theory of relativity, are to be interpreted directly as laws relating to the possible relative positions of solid bodies at rest; and, in a more general way, the laws of kinematics are to be interpreted as laws which describe the relations of measuring bodies and clocks. To two selected points of a stationary rigid body there always corresponds a distance of quite definite length which is independent of the locality and orientation of the body, and is also independent of the time. To two selected positions of the hands of a clock at rest relatively to the privileged system of reference there always corresponds an interval of time of a definite length, which is independent of place and time. We shall soon see that the general theory of relativity cannot adhere to this simple physical interpretation of space and time.[13]

From the time of Newton, the laws of physics had been written as mathematical equations in which there entered in geometric expressions involving position and time, and the physical interpretation of the mathematical descriptions of physics had always relied on the idea that the quantities represented by the variables of position and time directly corresponded to what one would get if one were to measure distances and time intervals using rulers and clocks. The rules for calculating relative positions and orientations of bodies in space were those of Euclid, and the presupposition was that our observations of nature would always correspond to the calculations made on the basis of Euclidean geometry provided only that these calculations are done correctly. Once one has abandoned belief in the existence of absolute space (and absolute time), so that all motions are relative, there is no other real physical significance to the idea of "space." One has only geometrical relationships among the mathematical coordinates of things. Thus it seemed that the theory of relativity had vindicated Leibniz' view of space over that of Newton.

However, the equations of physics do not stand in isolation from one another. The laws of physics do not take turns in applying to nature but, rather, all of them are to apply at all time to all their objects. Therefore, the equations in which they are written must not only describe the phenomena for which they are the description, but they must also be utterly consistent with one another under all circumstances. Now, there is nothing in the axioms of geometry that can guarantee this universal cohesion for the laws of physics, for the requirement that such a cohesion

[13] A. Einstein, "The foundation of the general theory of relativity," *Annalen der Physik*, 49, 1916.

must be maintained is not a law of mathematics but rather a requirement of the aim of physics itself. The role of mathematics in physics is *descriptive*. The doctrine of mathematics does not tell us what the laws of physics are to be; it merely tells us the rules that pertain to *mathematics* and to mathematical relationships among mathematical (intelligible) objects.

It was mentioned earlier that some of the Newtonian laws of physics had mathematical forms that changed when the geometrical system of coordinates in which they were described was changed. This change in the mathematical form, when interpreted in physical terms, did in some cases lead to differences in physical consequences. For example, under the Galilean velocity transformation it is impossible for the velocity of light to be "a universal constant." The special theory of relativity *had placed constraints on the mathematical forms* of physics equations such that the laws of physics *had to be expressed* in such a way that these equations were invariant when expressed in different coordinates expressing uniform relative motion of one observer with respect to another. The general theory of relativity carries this condition of constraint farther by requiring that, in Einstein's words, "The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion." This statement is the principle of general relativity.

> In classical mechanics, as well as in the special theory of relativity, the co-ordinates of space and time have a direct physical meaning. To say that a point-event has the X_1 co-ordinate x_1 means that the projection of the point-event on the axis of X_1, determined by rigid rods and in accordance with the rules of Euclidean geometry, is obtained by measuring off a given rod (the unit of length) x_1 times from the origin of the co-ordinates along the axis of X_1 . . .
>
> This view of space and time has always been in the minds of physicists, even if, as a rule, they have been unconscious of it. This is clear from the part which these concepts play in physical measurements . . . But we shall now show that we must put it aside and replace it by a more general view, in order to be able to carry through the postulate of general relativity, if the special theory of relativity applies to the special case of the absence of a gravitational field.[14]

Einstein was able to present examples in which the determinations of physical distances or lengths by means of measuring processes employing the rules of Euclidean geometry led to results that are inconsistent with the requirements of the special theory of relativity. What in effect this meant was: the presupposition that the rules of Euclidean geometry automatically apply in the description of physical events leads to contradictions. The inconsistencies that Einstein presented in his examples were shown to arise fundamentally from tying measurement procedures using "rods and clocks" to the rules of Euclidean geometry.

> We therefore reach this result: - In the general theory of relativity, space and time cannot be defined in such a way that differences of the spatial co-ordinates can be directly measured by the unit measuring-rod, or the differences in the time co-ordinate by a standard clock.

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[14] *ibid.*

> The method hitherto employed for laying co-ordinates into the space-time continuum in a definite manner thus breaks down, and there seems to be no other way which would allow us to adapt systems of co-ordinates to the four-dimensional universe so that we might expect from their application a particularly simple formulation of the laws of nature. So there is nothing for it but to regard all imaginable systems of co-ordinates, on principle, as equally suitable for the description of nature. This comes to requiring that: -
>
> *The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatever (generally co-variant)*.[15]

Let us observe that what Einstein is telling us is not a law about any ontological feature of material bodies or even of space and time as *things*. He is prescribing a *rule about rules*. Specifically, he is prescribing a constraint that must be applied when the physicist tries to express the observable behaviors of nature in term of mathematical equations. Just as Margenau pointed out certain particular features that an equation must have if it is to be interpretable in terms of physical causality, so Einstein is telling us about a constraint laid on the form of mathematical equations when these equations purport to describe geometrical relationships among objects. And why is this constraint laid upon our geometric description of nature? It is *because the laws of physics cannot be made to depend on arbitrary decisions about how geometry applies to the description of physical phenomena*. The geometry one uses is not a matter of convenience.

From the time of the Pythagoreans (the sixth century B.C.) until the mid-nineteenth century there had been only one kind of geometry, and that was Euclid's. The axioms of Euclidean geometry were regarded for centuries as the first, best example of self-evident truths. They were the fortress and citadel of rationalist philosophy. But these "self-evident" truths are nothing of the sort. They are rules reflecting common sense beliefs derived from such constructive acts as the drawing of a straight line, and from subjectively comfortable abstractions based on projecting from physical lines to infinitesimal lines "without breadth" and projecting from such lines to infinitesimal points "without length." Upon these abstractions is based the idea of measurements "by unit measuring-rods." But these conclusions from the axioms of Euclid are abstractions to the infinitesimal that are never met with in any possible experience and they lack the important feature of *necessity*. In the nineteenth century Riemann and others proved it was possible to deny some of Euclid's axioms, replacing them with other axioms, and still obtain mathematically consistent systems of geometries in which, for example, there is no such thing as "parallel lines extending to infinity," or in which "parallel" lines can intersect.

Consequently, an ontology in which "space" is regarded as a *thing* has no real foundation in the phenomenal world. As Einstein put it,

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[15] *ibid.*

> That this requirement of general co-variance, which takes away from space and time the last remnant of physical objectivity, is a natural one will be seen from the following reflexion. All our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of these points. Moreover, the results of our measurings are nothing but verifications of such meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place at the same time.
>
> The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences . . . As all our physical experience can be ultimately reduced to such coincidences, there can be no immediate reason for preferring certain systems of co-ordinates to others, that is to say, we arrive at the requirement of general co-variance.[16]

Einstein was able to deduce a system of differential equations that determine the requirements that must be met by the geometrical system of mathematical rules. These rules determine the so-called "metric space" by which the system of co-ordinates to be used in the mathematical expression of physical laws guarantees the property of general co-variance. Einstein's equations force the *description* of events to properly describe the motion of bodies that is attributed to the phenomenon of gravity. Gravity *per se* in the general theory of relativity loses its ontological interpretation as a "force" and becomes merely a *noumenon* that stands as a condition for tying the geometry of "space-time" to phenomena. Space is given no *ontological* interpretation. Its context is fixed to *formal* rules of relationships ("space-time") among physical things. Indeed, Einstein divorces "gravity" from "matter" (objects) altogether.

> We make a distinction hereafter between "gravitational field" and "matter" in this way, that we denote everything but the gravitational field as "matter." Our use of the word therefore includes not only matter in the ordinary sense, but the electromagnetic field as well.[17]

This is partially a vindication of Leibniz' view, but Einstein's theory goes well beyond Leibniz' theory inasmuch as it prescribes precise rules for the construction of a mathematical metric space, and it does so without any need to invoke Leibniz' monads or any ontological entity as a substratum for the idea of "physical space." Space-time is *the condition* of *coherence and consistency with possible experience in the mathematical description of Nature*.

When a physicist says, "space is curved," this statement can have no ontological moment, no objective validity with regard to any thing-like space. The statement has objective validity only insofar as it is taken to mean that the geometry for describing physical laws meets the requirements of mathematical form laid down by the Einstein equations. The idea of space as a "thing" with ontological properties of its own is without any objective validity whatsoever and, rather, is nothing more than the transcendent idea of a *noumenon*.

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[16] *ibid*.
[17] *ibid*.

Objective space therefore has objective validity only inasmuch as through geometrical rules the idea of space provides a constructive description for a manifold in reciprocal relationships among phenomenal objects. It has this validity only inasmuch as these geometrical rules meet with the constraint of general co-variance in the mathematical forms in which the laws of physics are expressed. In mathematical language, objective space-time is a metric space. *The Penguin Dictionary of Mathematics* defines "metric space" in general as

> **metric space**: A set of points is a metric space if there is a metric d which gives to any pair of points x, y a non-negative number $d(x,y)$, their distance (or separation), and is such that
> (1) $d(x,y) = 0$ if and only if $x = y$,
> (2) $d(x,y) = d(y,x)$, and
> (3) $d(x,y) + d(y,z) \geq d(x,z)$ for any points x, y, z of the set.

The general theory of relativity does not allow any arbitrary metric space to be used in the description of Nature, but only those metric spaces which meet with the principle of co-variance laid down by Einstein's equations. In non-relativistic physics, physics is made subservient to the rules of mathematics insofar as the consequences of its mathematical laws are concerned. With the theory of relativity, *mathematics is first made subservient to physics* in terms of the expressions allowable in the mathematics, *and then physics is bound to the mathematical consequences of this form of expression*. The relativity theory is a theory of reciprocity between mathematics and physics.

Einstein's theory is the only instance, at the time of this writing, where rules deemed necessary for the coherent and self-consistent description of Nature dictated to mathematics the form of mathematics that *must* be employed. It has long been a source of wonder to scientists and laymen alike that mathematics – which is an invention of the human mind – should describe the world, which is presumably indifferent to the intellectual whims of man. But this question is misinformed. All our scientific understanding is understanding of Nature, not the world. Coherence in a *system* of experience is our ultimate standard gauge for reasoning about Nature, and Einstein's theory is the first time that science, outside of philosophy, has mandated that the requirements of possible experience must dictate to mathematics the linkage between possible physical experience and *noumenal* mathematical theory. We can hope it will not be the last instance. *The objective validity of Einstein's theory is practical objective validity*, and its proper Standpoint is the judicial Standpoint of the Critical Philosophy.

Is the objective space we have described here the "space" Kant introduced in the Critical Philosophy? No, it is not. Kant's space is a *subjective* rather than an objective space, just as "time" in the Critical Philosophy is subjective time rather than objective time. Both terms are transcendental in Kant's philosophy, and we must next explore what this means.

§ 2. The Psychology of Space: Poincaré and Piaget

Kant's pronouncement that "space" was none other than "the pure form of intuition of outer sense" almost immediately produced a cornucopia of opinion as to what exactly this meant. Some saw this as an attempt to justify Newton's absolute space on philosophical grounds. Others saw it as a rationalist prescription, i.e. as a statement that the axioms of Euclidean geometry were innate ideas. Kant writings lend themselves to both points of view as a consequence of the limited and more or less repetitious discussions he devotes to this question. His writings rarely unambiguously set down the distinction we will draw here between objective space and the pure intuition of space, and this lack of a clear distinction easily leaves the impression, which is inconsistent with the rest of the Critical Philosophy, that by "space" Kant meant "geometry." The problem is further compounded by confusion, again arising from Kant's too-brief descriptions, between the meaning of "intuition" and that of "idea" (in the *Begriff* sense). For example, Paton interpreted (or, better, *mis*interpreted) Kant's meaning in the following way.

> Because space is a pure intuition, pure geometry is possible; because space is the form of intuition, pure geometry must apply to the sensible world.
>
> Every part of space and time (and therefore every geometrical figure) is also a pure intuition. It can be known in abstraction from given sensations [PAT1: 106].

Paton's first statement would be correct if he had said "pure *geometries* are possible" (rather than a singular "pure geometry"). The rest of his interpretation of Kant's meaning is misplaced because space, time, and geometry in the context in which Paton *uses* these words are not *forms* but, rather, ideas of supersensible objects. He thus passes unnoticed from subjective to objective space and time, and this is an error that merely calling them "intuitions" does not correct.

Paton's error is an error almost everyone makes upon reading Kant's *Inaugural Address* and *Critique of Pure Reason*. Paton's analysis is, in fact, fairer and less inaccurate than that of most commentators. Margenau's misinterpretation of Kant is much more seriously in error.

> Rarely has a physical theory received so persuasive and so careful a transcription into philosophic terms as has Newtonian mechanics. The transcription was Kant's epistemology. An examination of the latter doctrine therefore affords us an opportunity . . . to show in what respects the flight of modern physics beyond Newton has caused Kant's view to fail . . .
>
> . . . If the gist of [Margenau quotes from *Critique of Pure Reason*] is taken to be the assertion that conceptual space is not borrowed from immediate experience, modern physics can only be said to confirm it. But it denies the allegation that a determinate metrical space must already exist as a logical presupposition of experience. Indeed it strongly suspects Kant's tendency to unbridled generalization . . . [MARG: 144-145].

> The contents of [Margenau again quotes *Critique of Pure Reason*] clearly reflect the limits of eighteenth century geometric knowledge. Non-Euclidean geometries had not yet been discovered, and the extravagance of announcing theorems with apodictic certainty was natural in that day. All this has changed completely, and we may well dismiss [the previous quotation] as expressing views which have become factually false today . . . [If] Kant's final conclusion were replaced by a milder one, stating that conceptual space is not compounded from immediate experiences, it would be wholly acceptable [MARG: 148].

Margenau is one of many very gifted scholars who think Kant's space and objective space are one and the same, and who presume that, because Kant knew nothing at all about the revolution in geometry that was to come in the nineteenth century (true), for Kant "geometry" could only mean "Euclidean" (false). This overlooks the fact that the first non-Euclidean geometry was found and published in 1733 by Saccheri, and there is reason to think Kant knew about it. Martin writes:

> Saccheri started from one of the equivalent propositions; he assumed that the sum of the angles of a rectangle is less than four right-angles. To his great astonishment Saccheri saw that he could develop a long chain of consequences from this 'false' proposition . . . With this discovery the first non-Euclidean geometry had been found. One of the first to practice it was Lambert, the Berlin mathematician who was a friend of Kant. . . Nelson, Meinecke, and Natorp have shown conclusively that under the Kantian presuppositions it is not only possible but necessary to assume the existence of non-Euclidean geometries [MART: 17-18].

Indeed, Kant himself refers to "spherical triangles" in his *Prolegomena* [KANT2a: 81 (4: 285-286)] in a context that we will later see does not neatly fit with the presuppositions that Kant viewed "space" and "geometry" as synonymous in the Critical Philosophy or that Kant assumed Euclidean geometry to be the only one possible.

Most people are sighted (not blind), and the close relationship in our thinking between objective space and geometry has a tendency to produce an understanding of "space" that in adults is heavily linked to visualization. However, because Kantian space is the pure form of intuition of outer sense, vision is not a sufficient basis for understanding Kantian space because sight is not the only sensory modality of outer sense. Poincaré was one who recognized the implications inherent in this for understanding the idea of space, and we will start with him.

§ 2.1 Geometrical Space and Representative Space

Some German idealists and some neo-Kantians of the late nineteenth and early twentieth century interpreted Kant's transcendental aesthetic of space as saying that "space" and Euclidean geometry were one and the same. For a long while they resisted the new non-Euclidean geometries of Lobatschewsky and Riemann and insisted that Euclidean geometry was the only "true" geometry. Martin remarked, "The discovery of non-Euclidean geometries called up a storm of indignation among both mathematicians and philosophers, and nineteenth-century Kantians (and many even later) took part in the storm lustily" [MART: 18]. The neo-Kantians in particular have been called "positivists who ceased to be positivists," and, with a few exceptions such as Natorp, did much to promote the Euclidean geometry misinterpretation of Kant's system. Non-Euclidean geometries caused consternation among mathematicians because these geometries were the first breech in the wall of the last citadel of the rationalist-Platonic view of mathematics as the truest expression of man's ability to know his world *a priori* through pure thought. Poincaré was only too happy to take them all on.

> Most mathematicians regard Lobatschewsky's geometry as a mere logical curiosity. Some of them have, however, gone further. If several geometries are possible, they say, is it certain that our geometry is the one that is true? . . . Now, to discuss this view we must first of all ask ourselves, what is the nature of geometrical axioms? Are they synthetic *a priori* intuitions, as Kant affirmed? They would then be imposed on us with such a force that we could not conceive of a contrary proposition, nor could we build upon it a theoretical edifice. There would be no non-Euclidean geometry [POIN1: 48].

Here Poincaré makes a sharp, simple argument that demolishes the erroneous view that the pure intuition of space is a pure intuition of geometrical form. It ought to be noted, however, that to say "synthetic *a priori* intuition" is not the same thing as to say "pure *a priori* form of intuition." *A priori* means nothing else than "before experience." To say a representation is synthetic is to say it is synthesized, i.e. put together. We never have a *sensuous* experience with the ideal objects of geometry (or, for that matter, those of mathematics in general), so to say an axiom of geometry is a "synthetic *a priori* intuition" really refers to nothing else than a synthesis of productive imagination by which we "make sense" of a supersensible object; this is not pure *a priori* form of intuition but merely thinking. Also, Kant's space is not some collection of pre-set "cookie cutter" forms with which the *materia* of sensation is stamped (as we will later see).

But if neo-Kantian rationalism is wrong in granting status to Euclidean geometry as pure *a priori* form of intuition of outer sense, what is left for the axioms of geometry? Are they the findings of an experimental (i.e. positive) science? That view, too, is untenable.

> Ought we, then, to conclude that the axioms of geometry are experimental truths? But we do not make experiments on ideal lines or ideal circles; we can only make them on material objects. On what, therefore, would experiments serving as a foundation of geometry be based? The answer is easy. We have seen above that we constantly reason as if the geometrical figures behaved like solids. What geometry would borrow from experiment would be therefore the properties of these bodies . . . But a difficulty remains, and is insurmountable. If geometry were an experimental science, it would not be an exact science. It would be subjected to continual revision. Nay, it would from that day forth be proved to be erroneous, for we know that no rigorously invariable solid exists. *The geometrical axioms are therefore neither synthetic a priori intuitions nor experimental facts*. They are conventions. Our choice among all possible conventions is *guided* by experimental facts; but it remains *free*, and is only limited by the necessity of avoiding every contradiction, and thus it is that the postulates may remain rigorously true even when the experimental laws which have determined their adoption are only approximate. In other words, *the axioms of geometry* . . .

> *are only definitions in disguise*. What, then, are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true . . . One geometry cannot be more true than another; it can only be more convenient [POIN1: 49-50].

If we replace Poincaré's "synthetic *a priori* intuition" by "pure form of intuition of outer sense," his conclusion expressed above is not only correct but also in accord with the Critical Philosophy.

What, then, of the subjective and pure *a priori* intuition of space? If it is not identifiable as the axioms of geometry, what is it? How can we "make sense" out of it and thereby understand it? This question is quite difficult because we wish to describe something utterly primitive. We can only, to borrow from one of Santayana's phrases, seek to describe by a circumlocution that which is a transcendental primitive. In other words, we must put together an intelligible object, the properties of which capture the logical essence of the pure intuition of space. How do we do this? How do we make concepts for capturing what is and what is-not the logical essence of Kantian space? We will begin this leg of our journey with Poincaré, but we will take up with other guides before we reach our destination.

In what was, no doubt, a tongue-in-cheek jab at the neo-Kantians (and perhaps meant for Kant as well), Poincaré noted,

> It is often said that the images we form of external objects are localized in space, and even that they can only be formed on this condition. It is also said that this space, which thus serves as a kind of framework ready prepared for our sensations and representations, is identical with the space of the geometers, having all the properties of that space . . . In the first place, what are the properties of space properly so called? I mean of that space which is the object of geometry, and which I shall call geometrical space. The following are some of the more essential: -
>
> 1st, it is continuous; 2nd, it is infinite; 3rd, it is of three dimensions; 4th, it is homogeneous – that is to say, all its points are identical with one another; 5th, it is isotropic[1]. Compare this now with the framework of our representations and sensations, which I may call *representative space* [POIN1: 51-52].

Poincaré next carried through with a not-brief discussion of the outer senses of vision and touch. His discussion here is that of a mathematician employing a mathematician's ideas of "abstract space," which carries the following definition[2]:

> **abstract space:** A set of entities together with a set of axioms for operations on and relationships between those entities. Examples are metric spaces, topological spaces, and vector spaces.

Poincaré thus described vision in terms of a "visual space" and touch in terms of a "tactile space." He also pointed out the necessity of joining to them a third type of "space," namely "motor space" [POIN1: 52-60]. Visual space he described as having three dimensions, but the third

[1] Having the same properties in all directions. Poincaré's discussion predates Einstein's 1916 paper.
[2] *The Penguin Dictionary of Mathematics*, second edition.

(depth perception) is not isotropic with the first two dimensions. Additionally, the first two dimensions, which involve the processing of light as it falls on the retina, are not homogeneous. Furthermore, visual space is found to be not-infinite in extent and, Poincaré conjectures, a closer examination of it would probably reveal that it is not continuous. Thus, visual space as he envisions it is not at all congruent with geometrical space.

Tactile space gets short shrift in his discussion. He merely notes that it is even more complicated than visual space and "differs even more widely from geometrical space. It is useless to repeat for the sense of touch my remarks on the sense of sight." But tactile space leads him to the idea of motor space.

> But outside the data of sight and touch there are other sensations which contribute as much and more than they do to the genesis of the concept of space. They are those which everybody knows, which accompany all our movements, and which we usually call muscular sensations. The corresponding framework constitutes what may be called *motor space* [POIN1: 55].

The tie-in between Poincaré's motor space and James' kinæsthetic idea from Chapter 16 is probably obvious to you. Poincaré goes on to say,

> From this point of view *motor space would have as many dimensions as we have muscles*. I know it is said that if the muscular sensations contribute to form the concept of space, it is because we have the sense of *direction* of each movement, and that this is an integral part of the sensation. If this were so, and if a muscular sense could not be aroused unless it were accompanied by this geometrical sense of direction, geometrical space would certainly be a form imposed on our sensitiveness. But I do not see this at all when I analyze my sensations. What I do see is that the sensations which correspond to movements in the same direction are connected in my mind by a simple *association of ideas*. It is to this association that what we call the sense of direction is reduced. We cannot therefore discover this sense in a single sensation . . . Moreover, it is evidently acquired; it is, like all associations of ideas, the result of *habit*. This habit itself is the result of a very large number of *experiments*, and no doubt if the education of our senses had taken place in a different medium, where we would have been subject to different impressions, then contrary habits would have been acquired, and our muscular sensations would have been associated according to other laws [POIN1: 55-56].

We will see later what Kant had to say about the "sense of direction." We might also appropriately note at this point that Poincaré could have gone on to talk about such things as our "sense of balance" and numerous other basic yet high-level "senses" our experience takes in. Poincaré summarizes his main points regarding "representative space" as follows.

> Thus representative space in its triple form – visual, tactile, and motor – differs essentially from geometrical space. It is neither homogeneous nor isotropic; we cannot even say that it is of three dimensions. It is often said that we "project" into geometrical space the objects of our external perception, that we "localize" them. Now has this any meaning, and if so what is that meaning? Does it mean that we *represent* to ourselves external objects in geometrical space? Our representations are only the reproduction of our sensations; they cannot therefore be arranged in the same framework – that is to say in representative space. It is also just as impossible for us to

> represent to ourselves external objects in geometrical space . . . Thus we do not *represent* to ourselves external bodies in geometrical space, but we *reason* about these bodies as if they were situated in geometrical space. When it is said . . . that we "localize" such an object in such a point in space, what does it mean? *It simply means that we represent to ourselves the movements that must take place to reach that object* . . . When I say that we represent to ourselves these movements, I only mean that we represent to ourselves the muscular sensations which accompany them, and which have no geometrical character, and which therefore in no way imply the pre-existence of the concept of space [POIN1: 56-57].

Poincaré's point is that the character of "representative space" is of such an inhomogeneous, anisotropic, multi-dimensioned character that it is simply impossible for such a manifold of sensations to somehow be intuitively folded neatly into the orderly, regular, and limited character of geometrical space. It is a powerful argument, even making allowances for his lapse into psychological speculation near the end of his argument.

Do we "represent to ourselves the movements that must take place to reach the object"? We will take up that discussion momentarily; Poincaré was not a psychologist nor a neurobiologist, and we can without apology give small weight to his own explanation of how we are able to localize objects, and even of how it is we are able to come to have ideas of geometrical relationships. Furthermore, we must regard his arguments as being not complete, insofar as Kant's pure intuition of space is concerned, because Kant's space is the pure and *a priori* form of intuition of outer sense, and the outer senses include hearing, taste, and smell as much as vision, touch, and postural/locomotive sensations. Here it is enough to say Poincaré's view was: Through a slow process of learning how to make compensating movements, and learning to make correlations between our body sensations and observations of displacements, we come to form ideas of geometry and ideas of the location of objects. There is even a name in mathematics for the class of operations he thought were involved. They are called mathematical **groups**, and he made the hypothesis that group structure was something innate in our minds.

> The object of geometry is the study of a particular "group"; but the general concept of a group pre-exists in our minds, at least potentially. It is imposed on us not as a form of our sensitiveness, but as a form of our understanding; only, from among all possible groups, we must choose one that will be the *standard*, so to speak, to which we shall refer natural phenomena [POIN1: 70].

We will soon see, from experimental evidence gathered by psychology, that Poincaré was apparently not too far wrong in his conclusion, but he was a little short of being right.

§ 2.2 The Child's Conception of Space

To understand Poincaré's hypothesis, and why it is not entirely correct, we must understand what a mathematician means by the term "group." A mathematical group is a set of "elements" (call this set *G*) and an operation (which we will denote by the symbol $\bullet$) such that all the following properties hold:

> **1) closure –** for all pairs of elements *a* and *b* in the set *G*, the element $c = a \bullet b$ is also an element of set *G*;
>
> **2) associative property –** for all triplets of elements *a*, *b*, and *c* in the set *G* the operation $\bullet$ has the associative property, i.e. $a \bullet b \bullet c = (a \bullet b) \bullet c = \mathrm{a} \bullet (b \bullet c)$;
>
> **3) identity element –** *G* contains exactly one element *e*, called the identity element, having the property that for any *a* belonging to *G*, $e \bullet a = a \bullet e = a$;
>
> **4) inverses –** for every element *a* in set *G* there can be found exactly one element *b*, called the inverse of *a*, having the property that $a \bullet b = b \bullet a = e$. (Note that *e* is its own inverse).

A set *G* with operation $\bullet$ that has only property 1 is called a groupoid; if it has both properties 1 and 2 it is called a semigroup; if it has properties 1, 2, and 3 it is called a monoid.

Poincaré proposed that the set *G* was a set of sensations, excluding muscular sensations, which correspond to the movements of an external object; the operation $\bullet$ was a set of muscular sensations, called "displacements" and corresponding to movements of the subject's own body. He proposed that we come to know physical space and to invent geometry only through learned associations of the displacements necessary to compensate for external changes. He further proposed that our minds have a native ability to combine *G* and $\bullet$ such that they form a group.

Now, there is a problem of presupposition inherent in this hypothesis, which Piaget pointed out.

> But it is well to realize that, if we take the point of view of the subject and not merely that of a mathematical observer, the construction of a group structure implies at least two conditions: the concept of an object and the decentralization of movements by correcting for, and even reversing, their initial egocentricity. In fact, it is clear that the reversibility characteristic of the group presupposes the concept of an object, and also vice versa, since to retrieve an object is to make it possible for oneself to return (by displacing the object itself or one's own body) . . . It is obvious, therefore, that without conservation of objects there could not be any "group", since then everything would appear as a "change of state". The object and the group of displacements are thus indissociable, the one constituting the static aspect and the other the dynamic aspect of the same reality. But this is not all: a world with no objects is a universe with no systematic differentiation between subjective and external realities . . . By this very fact, such a universe would be centered on one's own actions, the subject being all the more dominated by this egocentric point of view because he remains un-self-conscious. But a group implies just the opposite attitude: a complete decentralization, such that one's own body is located as one element among others in a system of displacements enabling one to distinguish between one's own movements and those of objects.
>
> This being so, it is clear that throughout the first two stages [of sensorimotor intelligence], and even in the third, none of these conditions is fulfilled; the object is not constituted and the different spaces, and later on the single space that tends to coordinate them, remain centered on the subject [PIAG29: 124-125].

We have seen earlier that the child's separation of object from action develops slowly over time, the child's initial "sensational observable" being merely *Obs.OS*. Even acknowledging Poincaré's placement of the group structure outside the subject's capacity for receptivity and within its broader capacity for *Beurtheilung* (judgmentation) in general, the prerequisite condition for having a group structure is not met because the infant does not initially distinguish G and $\bullet$ separately. This cannot happen until the child forms that all-important division in concepts, thought as a real division, between the Self and the not-Self. It is true that Poincaré did not present an absolutist's picture inasmuch as he allowed that the group structure need only "exist potentially" in the thinking faculty. But all this conjecture is utterly empty unless it is possible for objective representations (in the Kantian rather than the Piagetian sense of the term) to be presented in sensibility. Sensation alone is not sufficient for this; the matter of sensation must take on a form, and this form, like the sensation, must come from within the perceiving Subject and cannot be imposed by a fictitious copy-of-reality. We cannot talk about "data of sensation" until sensational matter has a representative form because there can be no representation of any "given" (data) without both matter and form. But Poincaré was right inasmuch as the form given to sensation in intuition is not an *a priori* geometry of physical space.

Do we have any clues and evidence pointing to the characteristics for what this pure, *a priori*, and transcendental form of intuition for objects of outer sense might be? Piaget and his co-workers carried out an extensive program of research into this very topic. Their findings and evidence are presented in detail in three works: *The Construction of Reality in the Child* [PIAG2], *The Child's Conception of Space* [PIAG5], and *The Child's Conception of Geometry* [PIAG9]. Note that these titles do not say "the child's 'impression' of space" or "the child's 'impression' of geometry." Piaget et al. found that the child's ideas of physical space and ideas of geometry are constructed gradually over time, i.e. that they are "conceptualized" and are not in and of themselves "intuitive" in the sense of "being pure form of intuitions." Piaget and his co-workers studied the childish development of these ideas from birth through age twelve years. Peering ahead a bit to look at the flavor of their answer, Piaget and Inhelder remarked that,

> abstract geometrical analysis tends to show that fundamental spatial concepts are not Euclidean at all, but 'topological'. That is to say, based entirely on qualitative or 'bi-continuous' correspondences involving concepts like proximity and separation, order and enclosure. And indeed, we shall find that the child's space, which is essentially of an active and operational character, invariably begins with this simple topological type of relationship long before it becomes projective or Euclidean [PIAG5: vii].

Now, what do the terms "topological" and "topology" denote when used as technical and mathematical terms? Mathematicians define "topological space" in the following way.

> **topological space –** A set together with sufficient extra structure to make sense of the notion of continuity when applied to functions between sets. More precisely, a set X is called a topological space if a collection T of subsets of X is specified satisfying the following three axioms:
> (1) the empty set and X itself belong to T;
> (2) the intersection of two sets in T is again in T;
> (3) the union of any collection of sets in T is again in T.

The collection of subsets, T, is usually called "the topology of X." No doubt most readers will find this definition somewhat opaque since a great deal of training in mathematics is required in order to appreciate the implications of this definition. To quote again *The Penguin Dictionary of Mathematics*, topology is

> the study of those properties of geometrical figures that are invariant under continuous deformations (sometimes known as "rubber sheet geometry"). Unlike the geometer, who is typically concerned with questions of congruence or similarity of triangles, the topologist is not at all interested in distances and angles, and will for example regard a circle and a square (of whatever size) as equivalent, since either can be continuously deformed into the other. Thus such topics as knot theory belong to topology rather than to geometry; for the distinction between, say, a granny knot and a reef knot cannot be measured in terms of angles and lengths, yet no amount of stretching or bending will transform one knot into the other.

Topology is concerned with defining such things as "what points are in some sense 'neighbors' of a specific point *x*," and with providing a rigorous means by which we can define "neighborhoods" of points. To the topologist the inside of your stomach, when your mouth is open, is "outside your body" because a route can then be traced from the outside world to the interior of your stomach without cutting through your body. A point three feet in front of your nose and a point located inside your stomach are, in this sense, "neighbors."

Commenting on Poincaré's hypothesis, Piaget wrote

> There is a mutual dependence between group and object; the permanence of objects presupposes elaboration of the group of their displacements and vice versa. On the other hand, everything justifies us in centering our description on the genesis of space around that of the concept of the group. Geometrically, ever since H. Poincaré this concept has appeared as a prime essential to the interpretation of displacements. Psychologically, the group is the expression of the processes of identification and reversibility, which pertain to the fundamental phenomena of intellectual assimilation, particularly to reproductive assimilation or circular reaction.
>
> . . . From the point of view of intelligence . . . in contrast to that of perception, it is the problem of groups which remains primary. But it is necessary that we shall attribute the widest meaning to this concept for . . . it is possible, purely from our psychological point of view, to consider as a group every system of operations capable of permitting a return to the point of departure. Considered thus, it is self-evident that practical groups exist prior to any perception or awareness of any group whatever. They exist from the beginnings of postural space and, one might go so far as to say, even from the most elementary spatial and kinetic organizations of the living being. In this sense it is permissible to speak of the *a priori* nature of this concept; it merely attests to the fact that every organization forms a self-enclosed system [PIAG2: 100-101].

However, and this is an important point, this way of understanding the genesis of the child's conception of space is a point of view open only to the mathematically-minded observer. Put briefly, "the baby doesn't see things this way."

> In his famous analysis of the concept of space intended to show its origins in experience and in the very constitution of the human mind, Henri Poincaré considers as elementary the distinction between changes of position and changes of state. Among the changes presented in the external world some can be corrected by body movements which lead perception back to its initial state (for example turning the head to find an object which has passed before the eyes), others cannot; therefore the first constitute changes of position, the second changes of state. Thus, from the outset, according to Poincaré, this elementary distinction places the spatial in opposition to the physical and at the same time attests to the primitive nature of the concept of "group."
>
> But can one, like Poincaré, consider this distinction as primitive? . . . Our analysis of the development of object concept raises doubt as to the simplicity of these various questions . . . There is nothing to prove that sensorimotor adaptation to displacements immediately brings with it the concept of changes in position and, above all, there is nothing to prove that an activity, even if its constitutive operations proceed by groups from the observer's point of view, leads the subject to perceive displacements as such . . .
>
> In the first place, in order that a change of position may be distinguished from a change of state, the subject must be able to conceive of the external universe as being solid, that is, composed of substantial and permanent objects; otherwise the act of finding a displaced image would be confused, in the subject's consciousness, with the act of recreating it . . .
>
> In the second place, and by virtue of this very fact, in order that a change of position may be opposed to changes of state, the external universe must be distinguished from personal activity. If the perceived phenomenon and the acts of accommodation necessary for its perception were not dissociated, there could be no consciousness of the displacement . . .
>
> In the third place, as this last remark makes clear, to conceive of a change of position is tantamount to locating oneself in a spatial field conceived as being external to the body and independent of the action. It consists, therefore, in understanding that in finding the displaced object one displaces oneself as the observer localized in space, the displacement of the object and that of the subject being relative to each other . . .
>
> But as we have seen in the analysis of object concept, none of these three conditions is present during the first stages. Far from consisting of objects, the universe depends on personal actions; far from being externalized, it is not dissociated from subjective elements; and far from knowing himself and placing himself in relation to things, the subject does not know himself and is absorbed into things.
>
> With regard to the concept of "group" it therefore seems clear that, even if the subject's movements constitute groups from the point of view of the observer, the subject himself is unable to imagine them as such . . .
>
> Nevertheless, like Poincaré, we shall not hesitate to speak of groups to designate the child's behavior patterns to the extent that they can be reversed or corrected to bring them back to the initial point. The only objection to Poincaré's description is that he considered such groups as capable of being immediately extended in adequate perceptions or images, whereas in fact they remain in the practical state for a long time before giving rise to mental constructions [PIAG2: 102-106].

Now it needs to be clearly understood that when Poincaré and Piaget refer to "space" (unmodified by any adjective) they are referring to "physical" – that is to say, *objective* – space. Piaget's findings demonstrate in convincing fashion that an interpretation of Kant's pure intuition of space to mean an *a priori* intuition of an *objective* space is contrary to fact. In terms of epistemology, objective space *is an object* in the Kantian sense of the word "object." As we will gradually come to appreciate, the term "pure intuition of space" (*a priori* form of outer sense) will have to be regarded as **a capacity for organizing perceptions in such a way that the conceptualizing of any object, including objective space, is possible**. The transcendental question is: what is necessary for this possibility?

To examine this question we do well to begin with, so to speak, the "material at hand" the infant has at his disposal for carrying out acts of perceptual organization. In the first stages of life we can see that the infant's capacity to organize his world is utterly dependent upon wholly *practical* capabilities because "innate ideas" and "copies-of-reality" have been dispensed with as being wholly contrary to fact.

> The conclusion to which the analysis of object concept has led us is that in the course of his first twelve to eighteen months the child proceeds from a sort of initial practical solipsism to the construction of a universe which includes himself as an element. At first the object is nothing more, in effect, than the sensory image at the disposal of acts; it merely extends the activity of the subject and, without being conceived as created by the action itself (since the subject knows nothing of himself at this level of his perception of the world), it is only felt and perceived as linked with the most immediate and subjective data of sensorimotor activity. During the first twelve months the object does not, therefore, exist apart from the action, and the action alone confers upon it the quality of constancy . . .
>
> The history of the elaboration of spatial relations and of the formation of the principal groups exactly parallels the foregoing. At first there exists only a practical space or, more precisely, as many practical spaces as are predicated by the various activities of the subject, while the subject remains outside of space to the precise extent that he does not know himself; thus space is only a property of action, developed as action becomes coordinated . . .
>
> This transition from a practical and egocentric space to the represented space containing the subject himself is not an accident in the elaboration of displacement groups; it is the *sine qua non* of the representation and even of the direct perception of groups, for we shall see that it is one thing to act in conformity to the principle of groups and another to perceive or conceive of them . . .
>
> But one sees at the same time how much our analysis of the child's space perception is simplified by the parallelism between the process just indicated and the process of formation of object concept. Just as during the first weeks of life the object is confused with the sensory impressions connected with elementary action, so also at birth there is no concept of space except the perception of light and the accommodation inherent in that perception (pupillary reflex to light and palpebral reflex to dazzle). All the rest – perception of shapes, of sizes, of distances, positions, etc. – is elaborated little by little at the same time as the objects themselves. Space, therefore, is not at all perceived as a container but rather as that it contains, that is, objects themselves; and, if space becomes in a sense a container, it is to the extent that the relationships which constitute the objectification of bodies succeed in becoming intercoordinated until they form a coherent whole. The concept of space is understood only as a function of the construction of objects . . .
>
> In effect, an initial stage during which space consists of heterogeneous and purely practical groups (each perceptual bundle constitutes a space) corresponds to the first stages of object concept . . . There are groups in the sense that the child's activity is capable of turning back on itself and thus of constituting closed totalities which mathematically define the group. But the child does not perceive these groups in things and does not become aware of the entirely motor operations by means of which he elaborates them; hence the groups remain entirely practical [PIAG2: 97-99].

The quote just given is of fundamental importance for our purposes in this treatise. It is therefore worth a brief detour to explain more clearly this mathematical idea of a "group."

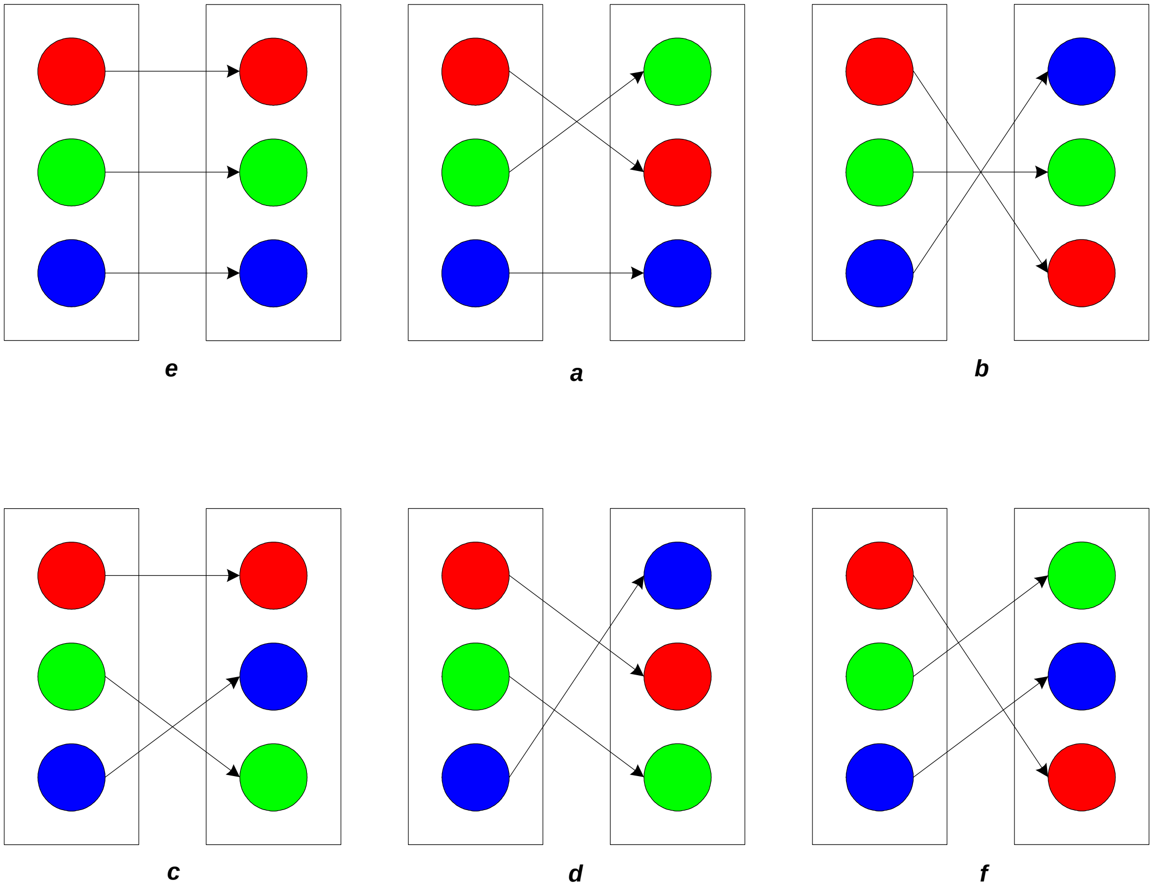

Figure 17.2.1: Illustration of a permutation group. Permutations *a* through *f* are denoted by the arrows illustrating the rearrangement of the balls from their initial to final positions. Permutation *e*, which leaves the positions of the balls unchanged, serves as the identity element of the permutation group.

§ 2.3 Analogy Example: The Permutation Group

What is meant by Piaget's explanation of the child's practical sensorimotor "spaces" in terms of "groups" can be illustrated by analogy using the example of a mathematical permutation group. Such a group is illustrated in Figure 17.2.1 above. A permutation is merely a rearrangement of the members of a set. In the case of this example we have a set consisting of three different-colored balls arranged in a column from top to bottom. The permutation consists of placing these balls in other arrangements. Symbols *a* through *f* denote the re-positionings and correspond to the arrows shown in the figure. It is the arrows that denote the members in the set *G* of the group and not the color of the balls. We can think of the arrows as being analogous to "movements," while the colored balls can be regarded as analogous to different sensational "images." It is important to understand that the group itself is entirely independent of the color of the balls and has to do merely with re-ordering an initial arrangement, shown as the left-hand box in each of the six cases above, into a final arrangement, shown as the right-hand box. For example, permutation *c* in the figure leaves the position of the top ball unchanged and swaps the positions of the middle and lower balls. So far as the permutation group is concerned, it does not at all matter what the color of the balls in the initial arrangement might be, nor what color they are in the final arrangement. All that matters is how the balls are repositioned. The permutations are "color blind."

The set *G* is the set of possible permutations, written as $G = \{a, b, c, d, e, f\}$. To form a group we must also have an operation, denoted by •, that operates on the members of *G*. In the permutation group this operation is *concatenation*, i.e. a succession of permutations. The operation $a \bullet b$ thus denotes performing permutation *a* on the initial arrangement followed by performing permutation *b* on the results of the first permutation. Referring to Figure 17.2.1, *a* produces the arrangement (green, red, blue) from top to bottom; applying *b* to this new arrangement produces (blue, red, green). We now take note that the outcome of this concatenation of permutations is equivalent to what we would have gotten by applying permutation *d* to the initial arrangement; therefore $a \bullet b$ is equivalent to permutation *d*. Symbolically, $a \bullet b = d$. We may next note that permutation *e* leaves the arrangement unaltered. Therefore for any permutation *x* belonging to *G* we will get $x \bullet e = e \bullet x = x$. Permutation *e* therefore is an "identity element" for the set *G*. Next, we note that some concatenations end up with the permutations canceling each other, i.e. $f \bullet d = e$ and $d \bullet f = e$. Thus permutation *d* is called the inverse of *f*, and *f* is likewise the inverse of *d*. In some cases a permutation can cancel itself, i.e. $a \bullet a = e$, thus *a* is its own inverse.

By taking the permutations in pairs we can construct an *operation table* for operation •. An operation table is similar to the addition table we all learned in elementary school. This operation table is shown in Figure 17.2.2 below. The table is arranged so that the order of the operations is from row to column, e.g. $a \bullet b = d$, etc. Referring to our earlier definition of a group, it is easily seen by inspecting the table that the closure property is satisfied. Inspection of the *e* row and the *e* column demonstrates that the identity property is likewise satisfied. Each row of the table has in it

•	*a*	*b*	*c*	*d*	*e*	*f*
a	*e*	*d*	*f*	*b*	*a*	*c*
b	*f*	*e*	*d*	*c*	*b*	*a*
c	*d*	*f*	*e*	*a*	*c*	*b*
d	*c*	*a*	*b*	*f*	*d*	*e*
e	*a*	*b*	*c*	*d*	*e*	*f*
f	*b*	*c*	*a*	*e*	*f*	*d*

Figure 17.2.2: Operation table for the permutation group

exactly one occurrence of *e*, and each column of the table has in it exactly one occurrence of *e*. This means that for each member in *G* there is exactly one member that acts as its inverse, and so the existence of inverses property of the group is satisfied. That the associative property is also satisfied is not so self-evident from the operation table, but this can be demonstrated by taking all triplets of permutations and applying the operation term by term across the triplet. For example,

$$(a \bullet b) \bullet c = d \bullet c = b; \qquad a \bullet (b \bullet c) = a \bullet d = b$$
$$\Rightarrow a \bullet b \bullet c = (a \bullet b) \bullet c = a \bullet (b \bullet c).$$

The same is found to be true for any triplet in *G* and so the associative property is established. This completes the proof that the permutation group is indeed a group.

The permutation group has another interesting property. If we extract from *G* the subsets

$$G_1 = \{a, e\}; \quad G_2 = \{b, e\}; \quad G_3 = \{c, e\}; \text{ and } G_4 = \{d, e, f\}$$

and apply Figure 17.2.2 to construct operation tables for these subsets, we find that each of these subsets by itself also constitutes a group (that is, each by itself satisfies the four properties that define a group). These four groups are called the *subgroups* of group *G*. Note that the union of these four subsets gives us the set *G*. Note further that the only things each subgroup have in common with the others are the inclusion of the identity element *e* and the operation •. The group *G* is not obtained simply as the union of the four subgroups because this union leaves undefined the results of operations such as $a \bullet d$ (note that this operation is not defined by any of the operation tables of the four subgroups). Each of the four subgroups is analogous to a particular Piagetian sensorimotor scheme.

If we are given only the four subgroups and asked to find the group for the set *G* we would not be able to do so using only the operation tables of the four given subgroups. Instead we would have to *construct* the entries that would be missing from the operation table for *G* by going back to Figure 17.2.1 and finding out what happens when we concatenate all the "missing" combinations such as $a \bullet b$. In other words, we would have to *try it* and "see what happens." This is analogous to Piaget's idea of *structuring* a system by means of assimilation and accommodation. In the process of building up group *G* we would be said to "coordinate" the four original subgroups, and this is analogous to Piaget's coordination of sensorimotor schemes. Finally, we may note that in many cases the operations in group *G* do not commute. For example, $a \bullet b = d$ but $b \bullet a = f$. Thus, the order in which the operations are applied makes a difference.

The permutation group is therefore an example of what mathematicians call a "non-commutative group."

§ 2.4 Piaget's Practical Spaces

In the last section it was said the permutations ("movements") were color blind. If we regard the color of the balls as analogous to different sensational matter in sensibility, the color-blind "nature" of the "movements" means that the "motor actions" (to continue the analogy) do not intrinsically depend on the details of the sensational matter. In Figure 17.2.1 we used an "initial" order of (red, green, blue) to illustrate the group, but we could have used any other initial order involving these three colors equally as well. We used the sensational matter in order to figure out the equivalences in the operation table, but not to describe the members of *G* or its subgroups.

From this point of view, the "motor actions" (permutations) in and of themselves do not involve the specific material details "in a perception" and in this sense can be called "practical" representative factors in a scheme. Piaget found that in order to describe the behaviors of infants in the early sensorimotor stages of life, he had to do so in the context of what he called "practical sensorimotor spaces." We will now examine this idea.

At the time when Piaget was first publishing his landmark studies, in the late 1940s and early 1950s, psychology subscribed to the idea of an innate perceptual geometry in explaining human perception. Simply put, Piaget's work overturned this viewpoint.

> According to currently accepted explanations of the perceptual process, every perceptual 'field', from the most elementary to the most highly developed, is organized in accordance with the same type of 'structure'. This organization is supposed to be of a geometrical character right from the start, quite apart from the effects of the laws of 'good gestalt', and to involve the immediate formation of perceptual constancies of shape and size. This would mean that at any age a baby could recognize the shape of an object independent of perspective, and its size apart from its distance. Thus there would be from the very outset a perception of relationships at once spatial and metric. If this hypothesis were correct it would only be necessary to call to mind the laws of spatial configurations in order to describe perceptual space.
>
> But we have already shown in the work referred to above[3] that the constancy of the shape of objects is far from being complete at the outset, since at 7 or 8 months a child has no idea of the permanence of objects, and does not dream of reversing a feeding bottle presented to him the wrong way round[4] . . . Since then we have shown, together with Lambercier, that as regards size constancy, great differences still persist between an 8-year-old child and an adult, whilst Brunswik and Cruikshank have demonstrated its absence during the first six months of existence. It is thus by no means absurd to suppose that perceptual relationships of a projective order (perspective) and of a metric order (estimation of size at varying distances) should appear later than these more elementary spatial relationships whose nature has first to be defined. It is also quite obvious that the perception of space involves a gradual construction and certainly does not exist ready made at the outset of mental development [PIAG5: 5-6].

[3] *The Construction of Reality in the Child.*
[4] Observation 78 in *The Construction of Reality in the Child.*

Yet, although the most elementary constitution of projective or metric geometry is found to be absent in the child during the first two stages of sensorimotor intelligence, it also appears to be the case that the earliest perceptions are not entirely lacking in structural form. Piaget and Inhelder go on to say,

> The first two stages of development are marked by an absence of coordination between the various sensory spaces, and in particular by the lack of coordination between vision and grasping – visual and tactile-kinaesthetic space are not yet related to one another as a whole. . .
>
> It is therefore necessary . . . to try to reconstruct the spatial relations which arise in primitive or rudimentary perceptions (e.g. in the exercise of the reflexes of sucking, touching, seeing patches of light, etc., and the earliest habits superimposed on these reflexes). But since these initial perceptions fail to attain constancy of size and shape, what sort of relations go to make up such a space? [PIAG5: 6].

Piaget and Inhelder were able to identify five relationships which appear to be elementary constituents of the baby's earliest perceptions. All five undergo extensive development and change as the baby ages. The five "elementary spatial relations" are *proximity*, *separation*, *relations of order*, *enclosure*, and *continuity*.

> The most elementary spatial relationship which can be grasped by perception would seem to be that of 'proximity', corresponding to the simplest type of perceptual structurization, namely, the 'nearby-ness' of elements belonging to the same perceptual field. . .
>
> A second elementary spatial relationship is that of *separation*. . . But once again, such a spatial relation corresponds to a very primitive function, one involved in the segregation of units, or in a general way, the analysis of elements making up a global or syncretic whole. . .
>
> A third essential relationship is established when two neighboring though separate elements are ranged one before another. This is the relation of *order* (or spatial *succession*). It undoubtedly appears very early on in the child's life . . .
>
> A fourth spatial relationship present in elementary perception is that of *enclosure* (or *surrounding*). . . On a surface one element may be perceived as surrounded by others, such as the nose framed by the rest of the face. . . But it is clear that although the relationship of 'enclosure' is originally a perceptual given, no sooner do the factors of 'proximity', 'separation', and various types of 'order' become organized than 'enclosure' undergoes a complex process of evolution, particularly as regards three dimensions. . .
>
> Lastly, it is obvious in the case of lines and surfaces there is right from the start a relationship of *continuity*. But it is a question of knowing in precisely what sense the whole of the perceptual field constitutes a continuous spatial field. For quite apart from the fact that the various initial qualitative spaces are not for a long time coordinated among themselves, it has not been shown in any particular field, such as the visual, that perceptual continuity retains the same character at all levels of development [PIAG5: 6-8].

There is of course no guarantee that this list exhausts the roster of elementary and innate perceptual capabilities present at birth or developing through maturation within a short time after birth. Nor does their presence necessarily imply that these perceptual primitives are to be found in every sensory modality, or even to the same degree in different modalities displaying them. For example, what sort of 'enclosure' could there be for the sense of smell? And how could the psychologist-observer tell if 'enclosure' were a characteristic of smell?

However, there is one important clue obtained from the character of these apparently elementary perceptual capabilities: They are constituents of a *topological* space.

> Generally speaking, it is also true that the basic perceptual relationships analyzed by Gestalt theory under the headings of proximity, segregation of elements, ordered regularity, etc. correspond to these equally elementary spatial relationships. And they are none other than those relations which the geometricians tell us are of a primitive character, forming that part of geometry called Topology, foreign to notions of rigid shapes, distances, and angles, or to mensuration and projective relations. To adopt the hypothesis that constancy of shape and size are not the direct outcome of rudimentary perception immediately brings perceptually primitive space back into line with what topology rightly regards as the basic data of geometrical construction [PIAG5: 8-9].

Let us note well the last sentence of this quote. Piaget et al. are not saying that the baby has any such clear idea as "oh, there's something enclosed here." Indeed, Piaget's theory would deny any such interpretation since it implies a "something" other than the perceiving Subject (the baby), and the infant is a practical solipsist who has not yet formed the division of Nature in terms of a Self and a not-Self. These perceptual primitives can imply nothing more than some sort of form capable of serving as a constitutive function for presenting "basic data" that in due course the baby can apply to *construct* geometrical concepts.

But this is not all. Let us recall that at this stage of development everything about the infant's observable behavior tells us that his cognitions do not go beyond Piaget's *Obs.OS*. If there is a job of constructing to be done, this construction can only be possible by means of the infant's basic motor actions, i.e. the construction must involve the whole of the sensorimotor scheme and not merely the sensory part of it alone. This is where the idea of a practical space enters the picture.

When we speak of a "practical" space, we refer to an organization of perceptions that is not due to objective cognitions but rather is an organization based upon motor actions and their effects in sensibility as registered by *affective* perceptions. Piaget noted,

> For recognition to begin, it is enough that the attitude previously adopted with regard to the thing be again set in motion and that nothing new in perception thwart that process. The impression of satisfaction and familiarity peculiar to recognition could thus only stem from this essential fact of the continuity of a scheme; the subject recognizes his own reaction before he recognizes the object as such. If the object is new and impedes action, there is no recognition; if the object is too well known or constantly present, the automatism of habit suppresses any opportunity for conscious recognition; but if the object resists the activity of the sensorimotor scheme sufficiently to create a momentary maladjustment while giving rise soon after to a successful readjustment, then assimilation is accompanied by recognition. The latter is only the realization of a mutual conformity between a given object and a scheme all ready to assimilate it. Recognition accordingly begins by being subjective before it becomes object recognition . . . In other words, recognition is at first only a particular instance of assimilation: the thing recognized stimulates and feeds the sensorimotor scheme which was previously constructed for its use, and without any necessity for evocation. . . In order that the recognized picture may become an object it must be dissociated from the action itself

> and put in a context of spatial and causal relations independent of the immediate activity [PIAG2: 6].

We can see at once how this finding accords with our theory, in which empirical intuitions owe their marking at a moment in subjective time to a reflective judgment of affective perception. In this early stage of sensorimotor intelligence the recognition of *Obs.OS* does not differentiate that in sensation which we would credit to vision, touch, etc. from that in sensation which corresponds to the "kinaesthetic idea" (to use James' term). In such recognition there is no objective space as such simply because one is not needed in order to mark a representation in sensibility as an intuition. What is required is a feeling of expedience in sensibility *and* a stimulation of a motoregulatory expression that puts in play the sensorimotor scheme.

The early reflexes and first habits of the infant, that behaviorally can be associated by the observer with external sensory modalities, are initially uncoordinated. For example, the infant initially does not coordinate vision with prehension, i.e. the "looking reflex" and the "grasping reflex" appear to operate independently of one another. An infant in the first few days of life will suck his thumb if the thumb accidentally comes into contact with his lips, but does not yet know how to bring the thumb to the lips on purpose. If he accidentally loses contact between the thumb and the lips he does not know how to re-establish this contact, and even appears not to realize that such re-establishment could be accomplished.

> To begin with the spatio-temporal structures, we observe that in the beginning there exists neither a single space nor a temporal order which contains objects and events in the same way as containers include their contents. There are, rather, several heterogeneous spaces all centered on the child's own body – buccal[1], tactile, visual, auditory, and postural spaces – and certain temporal impressions (waiting, etc.), but without objective coordination. These different spaces are then gradually coordinated (buccal and tactilo-kinesthetic, through sucking objects, for instance), but these coordinations remain partial for a long time, until the formation of the scheme of the permanent object has led to a fundamental distinction – which H. Poincaré wrongly thought was given from the outset – between changes of state, or physical modifications, and changes in position, or movements constitutive of space [PIAG15: 15-16].

Lacking even the concept of a Self distinct from the not-Self, the infant's perceptions might perhaps best be described as "sensorimotor pictures" recognizable only through feelings and realizable (in the sense of "being real" to the child) only in the context of the sensorimotor scheme. It is only an observer, and not the child, who can describe the variety of situations presented in experience in terms of multiple and heterogeneous "practical spaces."

> The first of all the schemes constitutive of the child's space is that which Stern has called *buccal space*. The displacements of the mouth in relation to objects for sucking or of objects in relation to the mouth accordingly constitute the simplest practical groups it is possible to observe in the child.

[1] Buccal means pertaining to the mouth or cheeks.

> In this respect we can distinguish three aggregates of events: displacements of the mouth in the search for the nipple, the reciprocal adjustment of thumb and mouth, and the adjustment of objects seized for the purpose of sucking. . .
>
> As revealed in their elementary simplicity, such movements are already arranged in groups of displacements if one adheres to the description of the behavior itself, that is, from the observer's point of view. . . In a general way, the group is constituted by every coordinated totality of displacements capable of returning to the point of departure and such that the final state does not depend on the route followed; the simple accommodations of the mouth and hands are in this category.
>
> But if the child thus acquires from the very beginning a sense *sui generis* of positions and displacements, of forms and dimensions, it is evident that for him, that is, from the point of view of his perception or his representation, such systems of displacements do not constitute groups . . . First of all . . . to him they are only more or less stable sensory images which extend his own effort at accommodation. In the second place, and by virtue of this very fact, he does not dissociate his own movements from those of the object or the movements of his mouth from those of his hand, and thus does not establish any relativity between them. Finally and most important, he does not locate either himself or his movements in the same space as that of the objects perceived; his own movements constitute for him an absolute which is foreign to space and not a system of displacements capable of being perceived or represented from without. On the whole, buccal space is a practical space which permits the child to rediscover positions, perform movements, adapt himself to forms and dimensions, but which does not at all allow him to apply such schemes beyond the immediate action [PIAG2: 106-108].

When we look at Piaget's other "practical spaces" we find the same general character. From the viewpoint of the observer, the child's various schemes are describable by an observer as being organized as groups, but the child himself has no objective knowledge of these group structures. They are, therefore, structures which are merely practical and not cognitive. It is this character that Piaget means by the term "practical group."

> In short, if perception of visual space involves the presence of practical groups, nothing warrants the assertion that the child perceives, or *a fortiori* has an image of, the displacements of objects in the form of groups; objects are not yet perceived either in their interrelations or in relation to the body itself conceived as mobile in space.
>
> The same is true of *auditory space, tactile space*, etc. If the child quickly learns to localize sounds (*O.I.*, obs. 44-49)[2], to find the relinquished object with his hand (*O.I.*, obs. 52-54 and see above obs. 4) etc., that does not prove in any way that he arranges perceived positions and displacements in groups; he is capable of following a displacement or of finding a position connected with his own attitudes but not of objectifying these factors in groups which are independent of the action. This can be said even more strongly of kinesthetic or postural space, that is to say, of the equilibrium of the body itself.
>
> In conclusion, two main aspects characterize these first two stages from the point of view of knowledge of spatial relations: the purely practical nature of the presenting groups and the relative heterogeneity of the different spaces.
>
> Each kind of space involves the existence of groups. . . But the child is not yet capable either of perceiving things in space in conformity to this group structure or, still less, of having an image of the groups thus formed; he puts the group into practice without having either direct or indirect knowledge of it . . .
>
> Moreover, these practical groups remain heterogeneous among themselves. As yet, no constant relation exists between the visual and buccal space or between tactile and visual space. True,

[2] *O.I.* refers to *The Origin of Intelligence in Children*, [PIAG1].

> auditory and visual space are already coordinated, as are buccal and tactile space, but no total and abstract space encompasses all the others. Hence each activity gives rise to an ordination *sui generis* of reality in space, but perceived spatial relations are not unified and, above all, there is no specifically geometric and kinematic representation that would make it possible to place them in a common environment [PIAG2: 112-113].

In regard to the last paragraph just quoted, namely in regard to Piaget's statement that "no total and abstract space encompasses all the others," it needs to be pointed out that the classification of different "spaces" given by Piaget is the observer's classification. Need it be said that the infant does not possess any idea of these classifications? Piaget's statement may be true so far as the observer is concerned, i.e. there is no observable "abstract space" that "encompasses" all the others. But this should not be over-generalized. In time the infant *does* come to coordinate his various practical spaces, he *does* come to distinguish *Obs.S* from *Obs.O*, he *does* come to form relationships among the objects of his perceptions, he *does* come to form concepts of an objective space, etc. However heterogeneous his practical spaces appear to the observer, none of these developments could take place *unless it were possible to find homogeneity among these heterogeneous practical spaces*. Some kind of homogeneous space is necessary for the possibility of assimilation of practical sensorimotor groups and the syncretic perceptual "pictures", whose presentation in sensibility makes assimilation possible, into a better-equilibrated overall structure. *It is this homogeneous and transcendental space to which the term "pure intuition of space" refers*. Howsoever different it may be from the various types of "spaces" we have looked at up to this point, the *functional* task of this pure and *a priori* form given to empirical representations in intuition is now clearer to see. **The logical essence of the pure intuition of space is: that in which homogeneity in the diverse practical spaces of sensorimotor schemes can be found**.

§ 3 Metaphysical Requirements of the Intuition of Space

We have just seen the factual and empirical backdrop against which the pure intuition of space (hereafter simply called "space") must operate. We now look at the metaphysical requirements placed on space that are necessary for objective representation in general. In the Critical Philosophy it is the judicial Standpoint that is the proper standpoint for a metaphysical exposition of these requirements. Here we encounter a difficulty when it comes to understanding these requirements by means of an interpretation of Kant's work. The difficulty stems from the absence within Kant's completed works of a full Critical treatment of the metaphysics of the judicial Standpoint. *Critique of Pure Reason*, the *Prolegomena*, and *Metaphysical Foundations of Natural*

Science are written primarily from the theoretical Standpoint. *Critique of Practical Reason*, *Foundations of the Metaphysics of Morals*, and *Metaphysics of Morals* are works which adopt the practical Standpoint. *Critique of the Power of Judgment* is the only one of the great Critiques written from the judicial Standpoint, and Kant never produced a corresponding "metaphysics of judgment" to accompany it. Palmquist has proposed that this was what Kant's unfinished work, known today as his *Opus Postumum*, was to have been [PALM3: 324-347], and his arguments in favor of this view command serious attention.

Why is the judicial Standpoint the proper standpoint for the metaphysics of space (and of time as well)? The judicial Standpoint is the Standpoint providing the joint between the theoretical and practical Standpoints. Space as a pure form of intuition has an obvious connection with the theoretical Standpoint (from which we understand our knowledge of objects). We have, however, just seen that there is a psychological dimension to the intuition of space as well, and that this psychological dimension works through the actions of the Subject. The metaphysics of action belongs to the province of the practical Standpoint. The exposition of space thus speaks to both Standpoints and must ultimately bridge them. Therefore, the metaphysics of space (and of time) belong to the judicial Standpoint.

From the theoretical Standpoint the pure intuition of space is a primitive. Consequently, it can have only functional descriptions in terms of objective exemplars in this Standpoint. Such exemplars (e.g. "physical" space, geometry, Piaget's multitude of "practical spaces" and sensori-motor groups, etc.) can give us some of the flavor of the pure intuition of space, but they cannot speak directly to the metaphysical requirements to be placed on our *Realerklärung* of space. To properly understand space ***as a function in receptivity***, we must first provide ourselves with an understanding of its transcendental place in Critical epistemology.

§ 3.1 Kant's Remarks in *Opus Postumum*

Kant began his discussion of space in *Critique of Pure Reason* with the question, "What is space?" From there he proceeded to briefly critique and dispose of both Newton's and Leibniz' theories of space. He followed this with a discussion of his own answer to the question, a discussion carried out from the theoretical Standpoint. But what is "space" regarded from the judicial Standpoint? In the *Opus Postumum* we find the following argument.

> That space and time are nothing in the Existing[3] *outside* the subject, much less still *inner* determinations of things, but merely things-of-thought (*entia rationis*)[4]:

[3] *Existirendes*. The connotation is that of *Existenz*. Kant is saying space does not externally attach to things.
[4] Rational things.

> What comes first is that space and time, and the object in them in indeterminate but determinable intuition – that is, in appearance – is given (*dabile*)[5] and so is thought as a possible whole (*cogitabile*)[6]. Both together, however, found a principle for synthetic *a priori* propositions . . . through which the subject constitutes itself into an object for physics; the latter does not introduce thorough-going determination from experience but for it, as a system of perceptions. – The subjective in intuition, as its form, is the object in appearance as it emerges *a priori* from synthetic representation according to this principle. The thing regarded as it is in itself is a thing-of-thought (*ens rationis*) of the connection of this manifold whole into the unity to which the subject constitutes itself. The object regarded as it is in itself = *x* is the sense-Object *regarded as it is in itself*, but as another mode of representation, not as another Object [KANT10: 180-181 (22: 414)].

There is much to untangle in this brief paragraph. First, the pure intuition of space (as well as the pure intuition of time) is *givable*; that is what Kant's "*dabile*" note means. How is it givable? By the thinking Subject to itself through the Subject's own power of receptivity. The pure form of space has its transcendental place in the power of sensible apprehension. The appearance given representation in sensibility constitutes an *a priori* proposition (axiom of intuition), and the pure form of intuition "founds a principle" for constructing the propositions of sensibility. Recall that the power of determining judgment judges the particular under a *given* general concept, but that it is beyond this power of judgment to make for itself these general concepts. It is in that sense – i.e. that the intuition marked at a moment in time will be made into a general concept in the synthesis of re-cognition – that an empirical intuition can be called an *a priori* proposition.

Second, we note that the pure form of intuition is not a determination *of* experience but rather a determination necessary *for* experience to be possible at all in the manner in which human beings come to *have* experience. This is the description of a *process* from which the appearance *emerges*. An emergent representation whose form is regarded as *givable* (*dabile*) rather than *given* (*datum*) is a representation we must regard as one that is *shaped* in the course of this process, and not as one that is pre-fixed as though the matter of sensation is given form like one gives form to cookie dough with a cookie cutter. That this is so is evident from Kant's statement that the determination of the pure form of the empirical intuition of an appearance *serves the making of a system of perceptions*.

Now elsewhere (e.g. in *Critique of Pure Reason*) Kant tells us that sensibility is not a judgment. The task of judgment, insofar as the making of a *system* of experience is concerned, falls to the power of reflective judgment. If the pure intuition of space were like some kind of

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[5] *Dabile* is one of Kant's more obscure Latin technical terms, and is probably best rendered as "the givable." Both *datum* and *dabile* stem from the Latin verb *do* (to give). The suffix –*ile* in Latin is used to denote a place where the objects of the noun to which it is suffixed can be found in abundance. Hence, *dabile* carries something of the connotation of a metaphorical stockroom or orchard where a *datum* (that-which-is-given) has its transcendental place. Thus *dabile* carries the flavor of a reference to a "that-from-whence-is-given," thus to the possibility of being given, hence "the givable."

[6] Conceivable.

sensational cookie cutter, what would there be for reflective judgment to judge? Would it merely be tasked with the selection of which cookie cutter to use? Such a tasking would be essentially *objective*, and the power of reflective judgment deals only with affective perceptions, and these only according to the principle of formal expedience. It makes far more sense to regard the process of receptivity in terms of a power of formal intuition that is *constructive*, i.e. a process that is capable of binding together the matter of sensation howsoever need be such that an equilibration between objective perception (intuition) and affective perception (feelings) can be *found*. The pure intuition of space is givable, not given, and this distinction implies *capabilities* rather than *prescriptions*. The subjective judgment of reflective judgment then has a task far easier to understand, namely that of accepting or refusing the offerings of the synthesis of pure intuition. In a famous court case one time the judge remarked, "I cannot define pornography, but I know it when I see it." So it is, metaphorically speaking, with reflective judgment as well.

> In the synthesis of intuition one cannot commence from empirical intuition with consciousness (from perception), for in that case the form would be missing. So one begins from an *a priori* principle of what is formal in intuition and proceeds to the principle of the possibility of experience: Draw nothing *from* experience and erect it [experience] yourself.
>
> All *Existenz* of consciousness in space and time is mere appearance of inner and outer sense, and, as such, a synthetic principle of intuition takes place *a priori*, and affects itself as a thing existing in space and time. The subject is here the thing regarded as it is in itself because it contains the spontaneity. Appearance is receptivity. The thing regarded as it is in itself is not another Object, but another mode for making oneself into an Object. The intelligible Object is not an *objectum noumenon* but the *act* of understanding which *makes* the Object of sensible intuition into a mere phenomenon.
>
> It is something in the *a priori* givable (*dabile*), that is, not a mere Object of intuition but intuition itself and not merely a thinkable object. It is not an *ens* (something in the Existing[7]) nor either a *non ens* (something unthinkable) but a principle of possibility.
>
> What is to be known through sense, i.e. perceived, must affect our sense, and the intuition of the Object which arises from it is appearance (thing regarded as it is in itself).
>
> Space is not something apprehensible (not an object of perception, i.e. of empirical representation with consciousness). Neither is it something in the *a priori* givable outside the thinking subject, but only an aggregate of representations which are in us; not something in whose idea there is a contradiction but which, however, is also not nothing [KANT10: 181 (22: 414-415)].

Kant is as much as telling us here that the objective validity of the idea of a pure intuition of space can only be a practical objective validity seen in terms of a principle of an act (namely, the act of forming a representation of sensibility). We are not given our experience through some fictional copy-of-reality mechanism, but rather we make our own experience as we go. That the form of the pure intuition of outer sense eventually leads us to posit an objective space is a by-product of this synthesis. Because pure intuition is at the service of the power of reflective judgment (for making a *system* of experience), we can assign its transcendental place only within

[7] *Existirendes*.

the thinking Subject itself. The pure form of intuition is givable, but the reception of what it has to offer is not determined by sensibility acting on its own accord. Rather, is determined by the formal expedience of the empirical intuition as judged by the process of reflective judgment.

Thus all our ideas of the pure intuition of space must suit a particular *context* in which the adjudicating principle is none other than the principle of formal expedience. The idea of the pure intuition of space does not require, and cannot require, such a sphere of *a priori* know-how as to predetermine a geometrical universe in advance of the acquisition of concepts of experience. Pure intuition does not in and of itself make geometric laws; it serves to make possible the *binding of sensible representations* such that appearances can fit within *one* overall system of experience. Its practical objective validity from the judicial Standpoint is to be judged in terms of the possibility of expedience in sensibility within the context of the Subject's general power of *Beurtheilung*.

For the infant who has only a meager acquisition of experience, this implies a great deal of possible diversity in the manner in which expedient sensibility can take form. But as the march of experience continues, it becomes more and more possible for different concepts to clash in the synthesis of reproductive imagination, and the possible forms of pure intuition that can be expedient within the system of experience become more and more constrained. For this reason we should expect the possible forms givable to empirical intuitions to take on more and more structure as experience builds up. In a manner of speaking, it is not imagination that is wrapped in Feynman's straitjacket; it is pure intuition that becomes so constrained with the building of the system of experience. (Hence arises the need for *teachers*; good teaching enhances intuition).

> The consciousness of myself is not yet an act of self-determination for the knowledge of an object, but is only the modality of knowledge in general by which a subject makes itself into an object in general, and is what is formal in intuition in general. Space and time, each of which is an absolute whole, together with the undetermined manifold, is the given fact (*dabile*) to which something else is set face-to-face as what is thinkable (*cogitabile*). – The representation as an act of knowledge is then called appearance, which contains an integration (*complexus*) according to the principle of bringing forth oneself [KANT10: 192 (22: 87)].

§ 3.2 Does Kant's Space Implicate a Euclidean Objective Space?

This point in our discussion seems like a good place to bring up and deal with what is perhaps the most serious issue involving Kant's pure intuition of space, namely whether it implicates a necessarily Euclidean objective space. In Kant's *Lectures on Metaphysics* we find the following.

> Now from this, that space and time are pure *a priori* intuitions and not concepts abstracted from Objects, it follows that all propositions of geometry and arithmetic, or that concern numbers and figures, are synthetic judgments, and indeed *a priori* and therefore necessary and apodictically certain, which would not be if they could not be derived from space and time as pure intuitions. E.g.,

> it can be found without experiment that one straight line is perpendicular at a given point [KANT19: 447 (29: 977)].

Have we not caught Kant red-handed? Is this not the smoking gun that proves the critics who interpret Kant to proclaim that space was what Margenau and Poincaré said he claimed? Granted that these words flowed from the pen of an attendee of Kant's lectures and not from the pen of Kant himself. However, these notes, known as *Metaphysik Vigilantius*, were not taken down by an ordinary youthful student of Kant's (see [KANT19: xxxviii-xxxix]), and Kant himself wrote words to more or less the same effect in his other works, e.g. in *Critique of Pure Reason* and the *Prolegomena*. Do these words not provide a clear example of what Margenau called "Kant's tendency to unbridled generalization"?

No, they do not. What these words prove is a tendency toward unbridled generalization on the part of Kant's critics. Let us take the preceding paragraph apart piece by piece and look at it. We start with the statement about the propositions of arithmetic and geometry being synthetic judgments. Of course this is true. Mathematics as defined by Kant is "knowledge obtained through the construction of concepts," and the construction of concepts always involves a synthetic judgment. Next, we note that these judgments are *a priori*. This is also true. We never have a direct experience with the ideal and supersensible objects of mathematics. *A priori* only means "prior to experience" and any synthetic judgment we make on a supersensible object is *ipso facto* going to be *a priori*. Now what about these *a priori* judgments being both necessary and apodictically certain? This is where the interpretation gets interesting.

First, let us understand that "proposition" of geometry or arithmetic means "theorem." Theorems are derived from the mathematical axioms. A theorem is necessary – that is, it is a necessarily true consequence of the axioms – only if it is provable and the proof is without error. We can therefore assume without losing the essence of the argument that "propositions" refers to correctly proved theorems and not to statements resulting from an erroneous mathematical deduction. But here note carefully that the only "apodictic certainty" we can impute to a theorem is that it is certainly *a true consequence of the axioms*. It is *not* an *a priori* and necessary and apodictically certain proposition of *physics*. This is to say that the theorem is true of the mathematical object; it can apply to the objects of physics only if the axioms apply to the objects of physics. It can be necessary for the objects of physics only if the axioms are necessarily true of the objects of physics. It can be apodictically certain for the objects of physics only if the axioms are apodictically certain for the objects of physics. But, as Kant pointed out many times, we know the objects of physics only as appearances, and the root concepts of the objects of physics are both empirical and contingent. Objective space, be it Newton's or Leibniz' space, is an object of physics and not an object of mathematics. The connection of the axioms of mathematics to the objects of physics is *not* an *a priori* judgment carrying necessity because the physical objects do not carry necessity in their composure *as things regarded as they are in themselves*.

The objective validity of the pure intuitions of space and time is merely practical, and the only necessity connecting them as constructive acts in the synthesis of apprehension is the necessity that they serve the purpose of constructing a *system* of experience. Empirical systems both can be and are modified constantly in the march of experience, in order that they remain systematic, and there is no warranty that my system of perceptions as it stands today will either last unaltered for all time or be immune from transcendent appendices in the realm of my ideas of supersensible objects.

Now, what of the statement that the propositions of geometry and arithmetic (i.e. of mathematics generally) cannot be *a priori*, necessary, and apodictic except for their being derived from the pure intuitions of space and time? It is absurd to take this statement to mean that the *axioms* of mathematics are laid down with apodictic certainty and necessity by the pure intuitions of space and time. This would be the same as to say that the pure intuitions of space and time are innate ideas, and Kant explicitly stated in numerous places in his works that there are no innate rationalist ideas *of any kind*. All knowledge *a priori* is "knowledge of know-how" – that is to say, knowledge of *acting*, not knowledge of objects.

The pure intuitions of space and time do, however, make it possible *to construct axioms*. The axioms of mathematics all deal with supersensible objects. Therefore they can never be *givens* from sensible experience. They are all constructs, i.e. "knowledge" (in the lay sense of that word) obtained through the construction of concepts. The axioms are made possible by pure intuition because without a means to build concepts of supersensible objects we could have no ideas whatsoever. In other words, without the pure form of intuition we could not make abstractions from sensible, empirical concepts to construct supersensible representations of the objects of ideas, and therefore could never have any axiomatic rules upon which to base the rest of mathematics. Axioms, as Poincaré said, are merely *definitions in disguise*, and all true definitions, in Kant's sense of the word 'definition', must be *a priori* or they could never stand as complete, precise, and delimited concepts (which the axioms of mathematics must be made to be).

> The synthesis of made concepts from which spring synthetic definitions is either that of *exposition* (of appearances) or that of *construction*. The latter is the synthesis of *arbitrarily* made concepts, the former the synthesis of empirically made concepts, that is, of concepts made out of given appearances as their matter . . . Arbitrarily made concepts are *mathematical* [KANT8: 142 (9: 141)].

An axiom is used as a fundamental proposition of mathematics.

> Fundamental propositions are either *intuitive* or *discursive*. The former can be exhibited in *intuition* and are called *axioms* (*axiomata*); the latter can only be expressed by notions and may be called *acroams* (*akroamata*) [KANT8: 117 (9: 110)].

In *Critique of Pure Reason* (B: 760-761), Kant called axioms "synthetic *a priori* principles insofar as they are immediately certain." He also there described them as "self-evident." These are statements that can have objective validity only through an interpretation from the judicial Standpoint. From this Standpoint, an axiom is "immediately certain" only in the sense that the intuition representing it in sensibility will be *believed* (not "known to be True" in the Hegelian sense) because teleological reflective judgments (which are forms of reflective judgments and deal with the connection of perceptions to the practical faculties of the Organized Being) are judgments of belief. In like manner, an axiom is "self-evident" only from the Standpoint of "being evident to the Subject" as appearances. To turn this perspective around, as easily happens when we adopt the theoretical Standpoint, is nothing less than to either assert a copy-of-reality hypothesis or invoke an innate idea. Belief is a holding-to-be-valid based on a merely subjectively sufficient ground. Beliefs can be overturned, also by reflective judgment, when they come into conflict with the synthesis of a system of Nature, and aesthetical reflective judgment is the instrument of their downfall.

How far, then, can we go in trusting the axioms of mathematics (and therefore mathematics itself) to apply *scientifically* to Nature? The answer is: Only insofar as the objects of the axiom are possible appearances and do not become transcendent. How is this to be determined? The answer is: By according to whether or not the axiom and its objects cohere with the acroams of the Critical Philosophy. Acroams deal with the transcendental, i.e. with what is necessary for the possibility of experience. To the extent that the axiom coheres with the acroams, it maintains a firm transcendental foundation and does not become a transcendent speculation. To the extent the axiom fails to adhere to possible appearance, it becomes transcendent and is no longer of a character necessary for the possibility of experience (and thereby becomes mere speculation).

To date there has never been a full "Critique of Pure Mathematics" aimed at determining mathematical axioms that systematically cohere with the acroams of the Critical Philosophy. I suspect most mathematicians will hate this next statement, but this means that even "pure" mathematics (as that term is used today) is an undiscriminated mix of objectively valid and speculative rules. Only Brouwer's[8] "constructivist" school of mathematics has made any attempt at such a critique, and I would say that their work falls well short of being a complete science.

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[8] Luitzen Egberton Jan Brouwer (1881-1966) was a Dutch mathematician and the founder of the doctrine of "intuitionism" in mathematics.

A consequence of this was the famous "crisis in the foundations" that swept the world of mathematics at the beginning of the twentieth century, and culminated in the resignation by the formalist school of mathematics of any claim that mathematics necessarily speaks to Nature or is a fountain of certainty. In his memoirs Bertrand Russell lamented,

> I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. But I discovered that many mathematical demonstrations, which my teachers expected me to accept, were full of fallacies, and that, if certainty were indeed to be discoverable in mathematics, it would be in a new field of mathematics, with more solid foundations than those that had hitherto been thought secure. But as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. Having constructed an elephant upon which the mathematical world could rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure than the elephant, and after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable.

As the saying goes, "A man's reach should exceed his grasp else what's a heaven for?" Russell began as a realist who became an empiricist who became a neo-Kantian who became a Hegelian who became again an empiricist, but an empiricist of the sort who deserves to be called in all honor a Russellian. Had he not fallen prey to the reversal of perspectives, which occurs when one searches in the theoretical Standpoint for what only the judicial Standpoint can provide with objective validity, who knows what more he might have accomplished?

To summarize where we are at this point, the pure intuition of space does not impute any kind of *innate* geometry, arithmetic, or other doctrine of mathematics to sensibility. Thus it also imputes no innate form of objective metric space. Doctrines of mathematics are construction projects and involve roles played by all the facets of *Beurtheilung* in general. But does Kant's theory tilt the scales so steeply that Euclidean geometry is thinkable while other geometries are "impossible"? Settling this issue is our next task.

§ 3.3 Does Kant's Space Deny Non-Euclidean Geometries?

Kant's critics often seize upon Kant's own examples to make their point. Earlier we noted that Vigilantius' lecture notes reflected what Kant had himself written elsewhere. Let us look directly at Kant's own printed words.

> Now space and time are those intuitions upon which pure mathematics bases all its cognitions and judgments which at the same time appear as apodictic and necessary; for mathematics must first present all its ideas in intuition, and pure mathematics in pure intuition, i.e. it makes them up[9], without which it is impossible for it to take a step as long as it lacks pure intuition (since mathematics cannot proceed analytically, namely through the analysis of concepts, but only

[9] Kant's phrase was *construiren*, another of his technical Germanized Latin terms.

> synthetically), in which alone the subject-matter for synthetic judgments can be given *a priori*. Geometry bases itself on the pure intuition of space [KANT2a: 79 (4: 283)].

This short statement is the lead-in for a number of Kant's examples that followed, and so it is worth our while to dissect what he is saying here. We will first note that there is something of an ambiguity in the first sentence stemming from Kant's reference to space and time as "those intuitions" rather than "those pure forms of intuition." The ambiguity is: does Kant mean the pure intuitions of space and time as the form of intuitions, or does he mean an intuitive representation of an objective space and an objective time? Taken in the context of what he had been saying just prior to the quote given above, it is most likely that he means the pure form of intuition. This is both because he is speaking of the possibility for mathematics to "make sense" out of its ideas, and because if he meant an idea of objective space or time he would be making what amounts to a psychological hypothesis about how mathematicians go about their toils. The latter, i.e. a psychological hypothesis, does not fit in the context of the point he is making, which is a discussion of the epistemological grounds for the possibility of pure mathematics (which deals with ideas of supersensible objects).

The next point to make is that Kant talks of those mathematical constructs which *appear* to be both apodictic and necessary. Most English translations of this passage from Kant's *Prolegomena* translate Kant's words in stronger terms, e.g. as "those cognitions and judgments which *come forth*" as apodictic and necessary. There is a subtle but nonetheless critical difference between a proposition that merely appears to be apodictic and necessary, and a proposition that *is* apodictic and necessary. The latter phrasing carries an implication of a truth that somehow goes beyond the mind of the mathematician and attains to the sort of Absolute Truth for which Hegel argued. We must here repeat that the truth of a mathematical proposition is a truth that follows relative to the axioms of mathematical thought, and that it is only relative to these axioms that such truths can be either apodictic or necessary. Only if we first believe the axioms can we hold to be certain the consequences of those axioms, and then we *must* so hold them to be certain. But if we doubt the axioms, then we may and must also doubt their consequences. That is the difference between a mathematical cognition that *appears* to be apodictic and necessary vs. one that *is* apodictic and necessary.

Next Kant speaks to the origin of mathematical objects. Mathematical objects are supersensible objects, and to have an idea of them we must make abstraction from sensible concepts. For example, we can "make sense" of the idea of a geometric "point" by starting with a spherical ball and then projecting what we think of it as this ball is made smaller and smaller until in an idealized limit its radius vanishes. At this point we have a "pure" intuition of a geometric point because we have taken away from our sensible object all those attributes that made the original ball-object a possible object of *actual* outer sense. We have taken from it all its sensational matter and left ourselves with an *idea* of an abstract form. This is what Kant means when he says pure mathematics requires pure intuitions. The thought *process* by which we accomplish this abstraction is one that is possible in thinking because of the pure *a priori* forms of intuitions of space and time.

Kant's comment that we "make up" the intuitions of mathematical objects refers to this abstracting process, and it should not be taken to imply that the mathematician engages in some sort of capricious fantasy, e.g. like a child "making up a story" for how his room got messy. Kant's word that has been translated here as "make up" was *construieren*, which is a Germanized version of the Latin word *construere*. This is Kant's usual practice when he wants to assign a limited technical connotation as a definition. *Construere* means in this context "to make by joining" and, less precisely, "to make into a heap." The "stuff" of supersensible mathematical objects comes from concepts of sensible objects which the mathematician must first join together ("make up into a presentation of an object") before engaging in the removal of those characteristics of these sensible objects that tie the mathematical object too much to the empirical contingencies of Nature. Put another way, mathematical objects serve a mathematical purpose and abstraction gets rid of those characteristics that do not serve this purpose. A geometric "line," for example, is not properly thought in terms of a huge number of "points" laid side by side; it is properly thought in terms of a process of abstraction where depth and width have been projected down to zero while length is retained (and possibly stretched out indefinitely in the case of an "infinite" line).

Understanding this process explains Kant's statement that mathematics proceeds synthetically rather than analytically (an assertion that has sometimes riled mathematicians). It is true enough that mathematical deduction often consists of taking apart mathematical ideas and examining them, and this is analysis; but before analysis can be applied to a mathematical idea we must first have the idea to be analyzed, and because all mathematical objects are supersensible, their ideas can only first come from synthesis. Advances in mathematics come from new ideas, new ideas require synthesis, and this is what Kant means when he says mathematics can only proceed (advance) through synthesis.

Finally, the processes of geometrical thinking, like those of mathematics generally, are active mental processes, and this is where we get the conclusion that the pure forms of intuition ground mathematics. The pure forms of intuition (space and time) are, as we have previously seen, capabilities required for mental *acts* in the synthesis of apprehension and of comprehension.

They are not static pre-formed structures analogous to a scaffolding used in erecting a building. They are *capacities for transformations* of matters of sensibility, go to active *structuring* of intuitive presentations, and would more properly have as an analog the carpenters rather than the scaffolding.

With all this in mind, let us now take a look at some of Kant's examples by which he attempts to explain the role of the pure forms of intuition by which pure mathematics, and more specifically geometry, is made possible. The first example has to do with geometric construction.

> In order to add something by way of illustration and confirmation, we need only to take a look at the usual and unavoidably necessary procedure of the geometers. All proofs of the thorough-going equality of two given figures (that one can be in all parts put in the place of the other) ultimately work out to this: that they are congruent with one another; which plainly is nothing other than a synthetic proposition based upon immediate intuition; and this intuition must be given pure and *a priori*, for otherwise the proposition could not carry weight as apodictically certain but would have only empirical certainty. . . That complete space (a space that is itself not the boundary of another space) has three dimensions, and that space in general cannot have more, is built upon the proposition that not more than three lines can cut each other at right angles in one point; this proposition can, however, absolutely not be demonstrated from concepts, but rests immediately upon intuition, and indeed pure *a priori* intuition, because it is apodictically certain; indeed, that we can demand that a line should be drawn *ad infinitum* (*in indefinitum*) . . . presupposes a representation of space and of time that can only attach to intuition, that is, namely, as it is not in itself utterly bounded; for this could never be concluded from concepts [KANT2a: 80-81 (4: 284-285)].

What is synthetic about the proposition of equality of two figures from congruence is the addition of the idea of congruence itself. Given two figures, there is nothing in the definitions of two such figures that includes the idea of "congruence." This idea is one which describes a certain relationship (namely the mathematical congruence relation) that applies to the figures when they are taken together. There is nothing in either figure, nor in just the two of them put together and overlaid, that contains the idea of congruence. The perception that they can be overlaid can, of course, be given empirically; but the idea that such a relationship implies complete equality of the two figures, and that this relationship will necessarily be obtained *every time* a congruence relationship exists, relies on pure intuition for the genesis of that idea of equality. Note, however, that it is only the possibility of the idea that pure intuition contributes. A child will not spot the relationship, and so it is not correct to assume that the idea of the equality of the figures is presented in intuition as a simple apprehension. The idea requires a synthesis of comprehension, i.e. the putting together of many concepts in sensibility to produce the idea.

Here it is also important that we clearly understand what Kant means when he says the proposition carries weight as "apodictically certain." In *Critique of Pure Reason* Kant states what the term "apodictically certain" means: combined with consciousness of necessity [KANT1a: 176 (B: 41)]. *Given* the premises (i.e. the concepts that go into the synthetic proposition), if one is conscious of the fact that the proposition must follow from the premises or a contradiction would arise, then (and only then) is the proposition apodictically certain. (Recall from Chapter 8 that "apodictic" is a logical momentum of Modality in judgment). A proposition is true (or certain) only in the context of its premises and not beyond or outside this context.

Kant's comment on space having three and only three dimensions probably seems timidly safe enough for those readers not burdened by training in advanced analytic geometry, but here we must also satisfy the mathematicians, physicists, and engineers who are so well-trained as to know about "n-dimensional" spaces, either with $n > 3$ (in many important technical theories) or even n as a non-integer (the by-now-famous "fractal geometry" of Mandelbrot). Is Kant saying that "space" must be exactly three-dimensional (and, by proxy, Euclidean)? No. Kant is talking about "the procedures of the geometers"; one must read a work such as Descartes' *Geometry* to really appreciate the gymnastics that geometers of that era carried out in describing curves and solid figures. The analytic geometry of today, based on "Hilbert space," seems almost trivial by comparison.

The very nature of these procedures led these geometers to confine themselves most faithfully to plane and solid figures that could be drawn or physically constructed. That limits the scope of the idea of "dimension" to lines that could be drawn at right angles, and no one at that time had a practical need for anything more than this. "Dimension" in eighteenth century solid Euclidean geometry does not mean the precise same thing as length, breadth, and depth (the Greek "dimensions" of "extension"), but is easily derived from Euclid:

> Def. 1 – A *solid* is that which has length, breadth, and depth.
> Def. 4 – A plane is *at right angles to a plane* when it makes right angles with all the straight lines which meet it and are in the plane.
> Prop. 4 – If a straight line be set up at right angles to two straight lines which cut one another, at their common point of section, it will also be at right angles to the plane through them.
>
> – Euclid, *Elements*, Bk. XI

By applying the procedure of proposition 4 above twice, three orthogonal lines are obtained which, taken in pairs, define three planes that can coordinate the description of any finite solid figure. Thus is "built" *the proposition* that geometrical space (as seen by those geometers) has exactly three dimensions. However, that these three planes can be extended without limit, and that therefore the "space" they define is unbounded, is not something that is contained in the construction but, rather, is a quality of the composition of this space stemming from the idea of a process of extending lines to infinity.

With regard to straight lines drawn "to infinity," we can note that Kant said we can "demand" this, and that such a demand must presuppose an intuitive representation of space and time, rather than an empirical one, because we never have any sensuous experience with "infinity" or the "infinitely large"; hence that we can "picture" this construction requires a formal capacity for intuition, without which we could never obtain a concept of anything unbounded.

Here, too, it is worth noting that even the idea of "distance" determined by a "straight line" is synthetic. This was a point Kant made in *Critique of Pure Reason*:

> That the straight line between two points is the shortest is a synthetic proposition. For my concept of *the straight* contains nothing of [extensive] magnitude but only a Quality. The concept of the shortest is therefore entirely additional to it, and cannot be abstracted out of the concept of the straight line by analysis. Help must here be gotten from intuition, by means of which alone the synthesis is possible [KANT1a: 145 (B: 16)].

With the benefit of a century of hindsight, we can note here that because "straight" is a particular Quality endowed to the idea of "line" its definition can be freely attached to different constructions of Quantity (aggregation of parts); mathematical definitions are "arbitrarily made concepts." Euclid defined "the straight line" one way, Riemann defined it in another way.

Kant's next example is one in which he manages to attack both Newton's and Leibniz' theories of space at the same time.

> All those who cannot yet get free of the idea, as if space and time were actual characteristics attaching to things regarded as they are in themselves, can exercise their acuity on the following paradox, and, if they have sought its solution in vain, can then, free of prejudice at least for a few moments, suppose that perhaps the demotion of space and of time to mere forms of our sensuous intuition might have foundation.
>
> If two things are fully the same (in all determinations belonging to magnitude and Quality) in all the parts of each that can always be known by itself alone, it should indeed then follow that one, in all cases and respects, can be put in the place of the other, without this exchange causing the least recognizable difference. In fact this is how things stand with plane figures in geometry; yet various spherical figures, notwithstanding this sort of complete inner agreement, nonetheless reveal such a difference in outer relationship that one cannot in any case be put in the place of the other; e.g., two spherical triangles from each of the hemispheres, which have an arc of the equator for a common base, can be fully equal with respect to their sides as well as their angles, so that nothing will be found in either, when it is fully described by itself, that is not also in the other, and still one cannot be put in the place of the other (that is, in the opposite hemisphere); and here is then after all an *inner* difference between the triangles that no understanding can specify as inner, and that reveals itself only through the outer relationship in space [KANT2a: 81 (4: 285-286)].

It is probably safe to assume that most of us do not spend much time contemplating spherical triangles, so let us make Kant's argument a bit more concrete with a specific example. Let us take an ordinary world globe and set it up so that 0° of longitude is directly facing us. Let one spherical triangle be placed with its base on the equator running from 60°W longitude to 120°W longitude. Its sides are the longitude lines running from the equator to the north pole. This triangle covers most of North America. Our second spherical triangle will have its base on the equator running from 60°E longitude to 120°E longitude with its sides likewise running up these longitude lines to the north pole. This triangle covers most of Asia.

In terms of their arc lengths on all three sides and all three of their angles, these spherical triangles are absolutely alike insofar as spherical Euclidean geometry is concerned. Yet from our point of view looking down at 0° longitude, triangle 1 bends to our right from its base to its tip, while triangle 2 bends to our left from its base to its tip. Although the two triangles are absolutely similar in terms of their "inner determinations," they exhibit (to us) a very pronounced difference in their external relationship to us. We cannot exchange them by means of mere translation (as is possible with plane triangles in Euclidean geometry) because if we do the apex of each triangle now points away from the earth and the triangles themselves contact the earth at only one point (90°W for triangle 2, and 90°E for triangle 1).

Now, why could we not "exchange" them another way? Let us merely rotate the globe until triangle 1 is on our right with its apex pointing to the left, and triangle 2 is on our left pointing to the right. That "exchanges" them, does it not? Well, no. For now instead of looking down at west Africa (0° longitude), we find ourselves looking down at the middle of the Pacific Ocean (180° longitude). We avoided a change in the external relationship of the triangles to each other *but not to us*. Kant said we must be able to exchange them without the least recognizable difference, and certainly this is a recognizable difference! In point of fact, this "exchange" is, from the viewpoint of relativity, a false "exchange" because triangle 1 still covers North America and triangle 2 still covers Asia. We are merely looking at the globe from the other side.

Kant's point is that by regarding the geometric properties as properties of space that "attach to things" a paradox is produced. We have here two things that insofar as inner determination goes are absolutely identical, but which are distinguishably different in geometric properties seen in external relationships. Newton's theory, with its absolute space, was prone to difficulties of this sort that were later used by Einstein to demolish Newton's view of space. The most familiar of these difficulties involved fictitious forces, called 'centrifugal forces', that had to be ascribed to circular motion in Newton's fixed absolute space. In the example above, we played the role of Newton's fixed absolute space. Leibniz' theory, although it denied absolute space and held all motions to be relative, also suffers from the paradox. Even though relative to the earth no exchange was effected, relative to the moon a difference in external relationship (which Leibniz also attached to things) is still obtained. If geometry is a property of space-as-a-thing attaching to things-in-space, the paradox is unavoidable: two geometrically identical things are at the same time geometrically different.[10]

In the particular example just quoted, all Kant is trying to do is illustrate the falsity of regarding geometric properties as being attached to things, i.e. space-as-a-thing to things-in-space independently of we who perceive external relationships among phenomena. However, this example affords the opportunity for a brief digression dealing with one of the propositions that has been made regarding whether or not Kant denied the possibility of non-Euclidean geometries. Martin has argued for an interesting middle ground in which he has Kant, in a sense, both affirming and denying the possibility of non-Euclidean geometries. We quote Martin in full:

> We are now in a position to clarify Kant's own attitude to Euclidean geometry. As we have already said, many Kantians have vigorously disputed the possibility of non-Euclidean geometry. This protest certainly has some ground in statements made by Kant, but the question is much more difficult than was at first assumed. It is made even more difficult by the fact that Kant – like Gauss later – avoided talking about non-Euclidean geometry, and in view of the conflicts that were kindled by the introduction of non-Euclidean geometries, we have to say that Kant was right to be careful. But there can be no doubt that it was clear to Kant that in geometry the field of what is logically possible extends far beyond that of Euclidean geometry. There was, however, one thesis to which Kant held firm – presumably in error. What goes beyond Euclidean geometry may be logically possible, but it cannot be constructed, that is to say, cannot be constructed in intuition, and this in turn means for Kant that it does not exist mathematically, that it is a mere figment of thought. Only Euclidean geometry exists in the mathematical sense and all non-Euclidean geometries are mere figments of thought. Kant gives a concrete example in his discussion of the Postulates.
>
> Thus there is no contradiction in the concept of a figure which is enclosed within two straight lines, since the concepts of the two straight lines and of their intersection contain no negation of a figure. The impossibility arises not from the concept in itself, but from the construction of the concept in space, that is, from the conditions and determinations of space. But these have their own objective reality, that is to say, they apply to possible things, since they contain in themselves a priori the form of experience in general (A 220, B 268).
>
> In terms of our present-day mathematical concepts this example is immediately intelligible to us. In Euclidean geometry two straight lines can never enclose a figure. But if the surface of a sphere is regarded as the model for a Riemannian geometry, then the great circles are Riemannian straight lines, and two straight lines can in fact make a figure (a lozenge). Yet until the question has been more closely investigated it must remain doubtful in what sense Kant envisaged a positive possibility of this kind. We know that Lambert saw the connection between non-Euclidean geometry and the surface of a sphere, and it would not be unthinkable for Kant to have taken this train of thought a stage further. But it is also quite thinkable that Kant proceeded in a purely conceptual way and considered whether a contradiction is contained in the pure concept of a closed figure composed of two straight lines as such. At all events, Kant reached the correct view in one

[10] It often seems to me that present day physicists tend to re-introduce the paradox that Einstein removed by putting geometry in its proper role with respect to objective space. Einstein's geometry justified itself by its placement with what we, as observers, will observe, i.e. by addressing what properties must go into our geometrical metrics determining the outcome of measurements of observable phenomena in objective space. In a sense, Einstein did for the mathematics of geometric quantity what Descartes did for Euclidean geometry; he provided a mathematics of "instruments" for geometric determination by observers according to their situation with respect to the observed. Many of today's physicists, even some very famous ones, speak of "geometrodynamics" in a tone that seems to endow space with thing-like properties of its own. And it seems to me that these properties become more occult every year. But, as Einstein said, "space is not a thing."

> way or the other and saw that the concept was not a contradictory concept. By this approach Kant took a bold step beyond Leibniz, since for Leibniz the concept of such a figure is contradictory.
> Kant thus denied the mathematical existence of a closed figure composed of two straight sides; it is indeed logically possible because it is free of contradiction, but it cannot be constructed and given in intuition. Kant's conception of non-Euclidean geometries in general has to be understood in the same way. Non-Euclidean geometries are logically possible but they cannot be constructed; hence they have no mathematical existence for Kant and are mere figments of thought [MART: 23-24].

Oh, really? Since Kant "avoided talking about non-Euclidean geometry" I wonder how Martin connected Kant's statement, that a figure enclosed by two straight lines *in Euclidean geometry* is impossible, with the conclusion that since this cannot be constructed in intuition *using Euclidean properties* it cannot be constructed in intuition using a non-Euclidean *definition* of what it is for a line to be "straight." Mathematical definitions are arbitrarily made concepts, and "straightness" is definable only as a Quality of composition in representation ("aggregation *A* is-not straight; aggregation *B* is straight"). If one chooses to define lines of longitude on a globe as "straight lines" then there is no problem whatsoever in constructing in intuition a figure enclosed by two such "straight lines." And, in fact, the surface of a Euclidean sphere does indeed serve as the model for the axioms of non-Euclidean geometries.

Martin's thesis contains a *saltus*, namely that the pure intuition of space is fundamentally geometrical, and that this geometry must be Euclidean. But this regards the pure intuition of space not from the judicial Standpoint (where this restriction is in no way called for) but from the theoretical Standpoint (where the objective validity of the idea of the pure intuition of space is not to be found). He also misses another key point. The pure intuition of space is *necessary for the possibility* of representation in intuition, and representation in intuition is *necessary for the possibility of knowledge through the construction of concepts*, i.e. mathematics. There can be no "figment of thought" that does not have mathematical existence in representation. Martin has a view of Kant that in effect has Kant saying we can think that which is unthinkable, and this is of course absurd. So far as I have yet been able to tell, Kant never positively affirmed the validity of non-Euclidean geometries, but neither did he positively affirm their impossibility.[11] And there is nothing in the grounds of objective validity for the idea of the pure intuition of space that closes the door on non-Euclidean geometry.

As a final example before concluding this section, we consider the following. In 1768, just a few years before his "Copernican revolution," Kant wrote a short essay entitled "Concerning the ultimate ground of the differentiation of directions in space" [KANT21: 361-372 (2: 375-383)]. In

[11] They would be "impossible" only if they could not be imagined without *mathematical* contradiction. This is quite different from saying they are impossible due to *ontological* contradiction. It was because of what he felt was an ontological contradiction that Kant's friend Lambert concluded non-Euclidean geometry was fallacious. Lambert did not find a mathematical contradiction [DAVI: 220].

this essay we can make out a Kant in transition. It is evident that Kant was already moving towards, and perhaps had already arrived at, an idea of an absolute space as something involving the makeup of the nature of human perception at its basis, but it also appears that this space was still basically a geometric space.

Kant has two main points to make in this essay. The first is that we know relationships in corporeal space only by reference to our own bodies:

> Because of its three dimensions, three planes may be thought in corporeal space, which intersect each other at right angles. Because all of us know what is outside us through the senses only in so far as they stand in reference to ourselves, it is no wonder that from the relationship of these intersecting planes to our body we draw the first ground to produce the idea of the direction in space [KANT21: 366 (2: 378-379)].

> Since the distinct feeling of the right and the left side is of such great necessity for judgment of direction, nature has immediately associated it to the mechanical arrangement of the human body [KANT21: 369 (2: 380)].

We, as adults at least, normally have no difficulty telling left from right, up from down, etc. Kant's thesis is that this simple ability could not be possible unless there is something in our own makeup from which we draw this "sense of direction."

His second point is far bolder. It is that this self-referencing capability, if we may call it that, constitutes an absolute and general reference, i.e. an "absolute space." My right side is on my right, no matter where I am, what "compass direction" I may be facing, or whatever other orientation my body may be in. Although I cannot perceive this absolute space as such, I can perceive differences in the relative placement of bodies in corporeal space. But these differences so perceived do not themselves establish for me a direction; this I do for myself. This absolute space is not an object of the outer senses but rather a *Grundbegriff* or "fundamental idea" that makes it possible for me to discern direction in corporeal space and from which I am able to form the determination of a corporeal *Gestalt*.

By the time of the publication of his *Prolegomena* fifteen years later, Kant had much refined this initial idea of the subjectivity of intuitive space.

> What indeed can be more similar to, and in all parts more equal to, my hand or my ear than its image in the mirror? And yet I cannot put such a hand as seen in the mirror in the place of the original; for if the one was a right hand, then the other in the mirror is a left, and the image of the right ear is a left one, which can never take the place of the former. Now there are no inner differences here that any understanding could merely think; and yet the differences are inner as far as the senses teach, for the left hand cannot, after all, be enclosed in the same boundaries as the right (they cannot be made congruent), despite all the reciprocal equality and similarity; one hand's glove cannot be used on the other. What then is the solution? These objects are surely not representations of things regarded as they are in themselves, and as sheer understanding would know them, but rather they are sensuous intuitions, i.e. appearances, whose possibility rests on the relationship of certain and as such unknown things to something else, namely our sensibility. Now space is the

> form of our outer intuition of this, and the inner determination of any space is possible only through the determination of the outer relationship to the whole space of which the former is a part (the relationship to outer sense); i.e., the part is possible only through the whole, which never occurs with things regarded as they are in themselves as objects of mere understanding alone, but well occurs with mere appearances. We can therefore make the difference between similar and equal but nonetheless incongruent things (e.g. oppositely spiraled snails) intelligible through no concept alone, but only through the relationship to right-hand and left-hand, which goes immediately to intuition [KANT8a: 81-82 (4: 286)].

Again, Kant is primarily arguing the case here for regarding space as a pure intuition rather than something outside ourselves and attaching to outside objects as things. The idea of having as part of that pure intuition a "sense" of left-handedness or right-handedness enters into the argument only in a peripheral way. However, this "self-referencing" aspect of the determination of incongruities in otherwise identical inner determinations of objects goes to the central part of the idea of the pure intuition of space as constituting a complete and absolute space that serves as the ultimate standard of reference for all determinations of *objective* spaces.

It should be obvious that this absolute space of intuition is in no way the same as Newton's absolute space. Kant has also dropped all references to geometry here. Nor is direct knowledge of this space, as an object, required in order to make determinations of objects of outer sense. But this intuition of an absolute space *is* necessary in order to be able to form first, in an intuition, a representation of external Relations among appearances, and second, in understanding, all concepts of "spaces" in a unity of consciousness *as a system*.

Poincaré rightly objected to the idea of a "geometrical sense of direction imposed on" motor space. If this "sense of direction" of which Kant speaks were "geometrical" then that would implicate the intuition of space as the intuition of a geometry, and all the objections of Kant's critics would apply full-force. But note that Kant is not talking about a "geometrical sense" of direction; all he is saying is that we make out a difference and that *from* the ability to make out a difference we can come to the *concept* of direction in an objective space. This is what Kant means when he says "the relationship to right-hand and left-hand goes immediately to intuition." The matter of this "sense of direction" is kinæsthetic, but its form is not geometric.

To sum up this section: Kant's pure intuition of space is seen as a capacity for the formation of a *Gestalt* in intuition. It is non-geometrical, although it is necessary for the possibility of making a geometry. In and of itself it prefers no particular system of geometry, and therefore is not prejudicial to the possibility of non-Euclidean geometries. Summing up in a broader perspective, the objective validity of the idea of the pure intuition of space is a practical objective validity to be viewed, in terms of metaphysics, from the judicial Standpoint and not from the theoretical Standpoint. The ground of its validity lies with the connection of actions to perceptions, and thus its idea belongs to that of a physical manifold (in terms of the representation of objects) conjoining motoregulatory expression with the sensuous capacities of receptivity. In terms of representation in general, the idea of the pure intuition of space goes to the transitive in Relation in the manifold of the sensorimotor capabilities of the Organized Being.

The metaphysic of space we have discussed here does not find itself in conflict with the findings of Piaget's research. Piaget tells us that the child's conception of geometry is a long time in coming. From the uncoordinated initial stages of practical scheme-groups, the child develops a series of better-equilibrated sensorimotor/cognitive structures, and over a long development these lead first to the child's earliest ideas of a projective geometry, and later to ideas of a metric geometry. This metric geometry is shown to be Euclidean, but this Euclidean character of the child's *objective* space is in no way *a priori*. It is rather the product of experience, hence *a posteriori*. The pure intuition of space makes this development possible, but it equally makes possible (through training in mathematics) the development of other objective and non-Euclidean geometries and spaces as well.

§ 4. The Pure Intuition of Space in the Theoretical Standpoint

Although metaphysically we can take only the judicial Standpoint in understanding Kant's pure intuition of space, in order to understand its specific characteristics we must adopt the theoretical Standpoint. This is because we need to establish what constitutes the nature of the rules by which empirical intuitions are represented, and thus we must be able to describe the relationship between the pure intuition of space and the representation of objects. As we discuss these characteristics, it is of utmost importance that we bear in mind that the pure intuition of space is the pure intuition of outer sense, and thus cannot be limited to only, say, visual or auditory or tactile senses.

When speaking from the theoretical Standpoint, Kant had more to say about what space is not than about what it is. This is usually a disappointment to those reading *Critique of Pure Reason* or Kant's *Inaugural Dissertation* for the first time, but we are now in a position to appreciate why this is so. Viewed objectively the pure intuition of space is a transcendental primitive, and this severely limits what we can say about it with theoretical objective validity. We can, to be sure, work on characterizing it in terms of its *function*, but most such characteristics are going to be empirical. With psychology not yet being a true science in his day, there was little material upon which Kant could draw for empirical characterizations of space in the eighteenth century.

Before we look at the principal theoretical characteristics of the *functional concept* of space, let us look at Kant's summary of his two main conclusions.

> a) Space imposes no property at all of any things regarded as they are in themselves, nor any relationship of them to each other, i.e. no determination of them that attaches to objects themselves and that would remain even if one were to abstract from all subjective conditions of intuition. For neither absolute nor relative determinations can be intuited prior to the *Dasein* of the things to which they pertain, thus be intuited *a priori*.
>
> b) Space is nothing other than merely the form of all appearances of outer sense, i.e. the subjective condition of sensibility, under which alone outer intuition is possible for us. . .
>
> We can accordingly speak of space, extended beings, and so on, only from the human standpoint. If we depart from the subjective condition under which alone we can come to outer intuition, namely that through which we can be affected by objects, then the representation of space signifies nothing at all. . . Our expositions accordingly teach the *reality* (i.e. objective validity) of space in regard to everything that can come before us externally as an object, but at the same time the *ideality* of space in regard to things when they are considered in themselves through reason, i.e. without taking account of the constitution of our sensibility. We therefore assert the *empirical reality* of space (with respect to all possible outer experience), though to be sure its *transcendental ideality*, i.e. that it is nothing as soon as we leave aside the condition of the possibility of all experience, and take it as something that grounds the things regarded as they are in themselves [KANT1a: 176-177 (B: 42-44)].

This is Kant's resolution of the argument concerning space between empiricism (Newton) and rationalism (Leibniz, Descartes). Space is no independent thing at all (other than in that connotation where we speak of a capability as being a kind of object) but rather an inherent capacity for making intuitive representations in sensibility. Note Kant's remark that there can be no determination of objects prior to the *Dasein* of these objects. Kant is not talking here about things coming into being in some mystical sense of creation. By "prior to the *Dasein*" of an object he means prior to the Organized Being coming to represent that an object exists in the *Dasein* connotation of "exists." The pure intuition of space provides the *Gestaltung* (formation) of the *Existenz* of an object as an object of outer senses.

Piaget's findings would seem to confirm this aspect of the pure intuition of space inasmuch as he found that the infant's conception of objects arises concurrently with his earliest sensorimotor acts that work to assimilate Piagetian objects into schemes. Space is not an occult "reality" in itself, but is rather an instrument of reality, i.e. a process presenting in sensibility the outer form of representations in appearances as the objects will be known to the thinking subject. I repeat again: nothing is real *to me* if I do not have a concept of the object connected with other concepts that give it a context. This means that nothing can be a substantial object for me unless there is some kind of meaning connected to it in my consciousness of the object. Empirically all meanings are ultimately based on actions, and seen in this way the pure intuition of space is an instrument for the assimilation of representations through actions. Kantian space is *subjective* space and so can be regarded only "from the human standpoint."

Kant listed four considerations that must be applied to the understanding of space from the theoretical Standpoint.

> 1) Space is not an empirical concept taken from outer experiences. For in order for certain sensations to be related to something outside me (i.e. to something in another place in space from that in which I find myself), thus in order for me to represent them as outside and next to one another, thus not merely as different but as in different places, the representation of space must already be their ground. Thus the representation of space cannot be obtained from the relationships of outer appearance through experience, but this outer experience is itself first possible only through this representation [KANT1a: 174-175 (B: 38)].

This consideration is a "negative" consideration; that is, it speaks to what space is not. It is a common error in the study of Kant to subvert this negative statement of limitation into a positive context, i.e. to make the *saltus* that Kant must be talking about geometry and, hence, the pure intuition of space must be an intuition of geometry. But this is not what Kant says and we must not over-generalize. The only "positive" statement he makes here is that there must be a capacity in receptivity for making representations in such a way that it is possible (eventually) to represent objects as appearing in different places. In no way does he say or imply that such a representation is immediately given upon the first presentation of an appearance. In terms of our general 2LAR of representation, Kant speaks here of space in terms of transitive Relation, i.e. of a capacity in sensibility common to all representations of objects. This capacity serves the logical momentum of disjunction in the construction of the manifold of concepts through the interplay of determining judgment and the synthesis of comprehension. It applies as much to smell or taste as to sight.

> 2) Space is a necessary representation *a priori* that is the ground of all outer intuitions. One can never make a representation that there is no space, though one can very well think that there are no objects to be encountered in it. It is therefore to be seen as the condition of the possibility of appearances and not as a determination dependent on them, and is an *a priori* representation that necessarily grounds outer appearances [KANT1a: 175 (B: 38-39)].

Here Kant does make a positive assertion with regard to space, but it is an error to let ourselves be seduced by his passing reference to objective space (the "space" in which no objects are encountered). This positive statement, in terms of the general 2LAR of representation, amounts to saying that the pure intuition of space is the *determining factor* in the Modality of representations of appearances in sensibility insofar as the appearance is presented in terms of *Existenz* in the composition of sensations. It serves the apodictic logical momentum of judgment in the construction of the manifold of concepts as concepts are brought into comprehension.

> 3) Space is no discursive or, as we say, general concept of relationships of things in general, but is a pure intuition. For, first, one can only represent a single space, and if one speaks of many spaces, one understands by that only parts of one and the same unique space. These parts cannot as it were precede the single all-encompassing space as its components (from which its composition would be

> possible), but rather are only thought *in it*. It is essentially single; the manifold in it, thus also the general idea of spaces in general, rests merely on restrictions. From this it follows that in respect to it an *a priori* intuition (which is not empirical) grounds all concepts of it. Thus also all geometric fundamental propositions, e.g. that in a triangle two sides together are always greater than the third, are never derived from general concepts of line and triangle, but rather are derived from intuition and indeed derived *a priori* with apodictic certainty [KANT1a: 175 (B: 39)].

In making this third point Kant throws us something of a head-fake by using objective space as an example of an object that illustrates the universal Nature of the pure intuition of space. The key point in the example is that ideas of objective spaces are always regarded in such a way that they appear as limitations on a "bigger" objective space (i.e. they are always subspaces).

There was recently an entertaining article written by Veneziano[1] discussing "string theory." String theory is one of present-day physics' hypotheses for trying to unite all the laws of physics under a common model. This particular article, among other things, presented a model of the universe in which "strings" lead to a splintering of the universe into numerous "black holes," within each of which forms a self-contained "universe." The theory would have it that *our* universe is merely one of a numberless many, each cut off from the others by virtue of residing inside a black hole. Among other things, this hypothesis attempts to speak to the "big bang" idea, but does not require the spontaneous creation of all the matter of the universe by a "quantum fluctuation." The relevance of this entertaining speculation to our discussion here is that in order to visualize these "black hole universes" we must embed them in an even larger objective space, one which contains all the black hole universes and, as well, such additional "string matter" as have not yet condensed enough to form their own black hole universes. (In this theory, "the" universe – the one comprised of black hole universes plus uncondensed string matter – has always existed). Presumably this "mother universe" is of indeterminate and undeterminable size. The article is an example of Kant's point that we cannot visualize any "space" without embedding it in an even larger "space."

Space as a pure intuition, on the other hand, must be regarded as a substratum, i.e. the perceiving Subject has only one pure intuition of space, and all concepts of objective spaces are developed by the placement of *restrictions* on the form of sensible representations of appearances. The idea of the Quality of the pure intuition of space from the theoretical Standpoint stands as an intensive magnitude, and the logical *momentum* of judgment in forming concepts to describe it is the infinite in Quality. Our idea of its appearance, when we seek to objectify it, falls under the category of limitation.

Finally,

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[1] G. Veneziano, "The myth of the beginning of time," *Scientific American*, pp. 54-65, May, 2004.

> 4) Space is pictured as an infinite *given* magnitude. Now one must, to be sure, think of every concept as a representation that is contained in an infinite aggregate of different possible representations (as their common mark), which thus contains these *under itself*; but no concept, as such, can be thought as if it contained an infinite aggregate of representations *within itself*. Nevertheless space is so thought (for all the parts of space, even to infinity, are simultaneous). Therefore the original representation of space is an a priori *intuition*, not a *concept* [KANT1a: 175 (B: 39-40)].

The pure intuition of space is described here in terms of a ground for the representation of *extensive* magnitudes. Kantian space is not built up by successive conglomeration of subspaces; rather, subspaces are represented by placing restrictions or limitations on what is represented in sensibility as an empirical intuition. However, Kant tells us, if there were not boundlessness in the progress of intuition no concept of relationships could bring to us even a subjective principle of induction to mathematical infinity [KANT19: 332-333 (28: 569)].

This presents us with an interesting interpretation of the word “infinity” Kant uses to describe space. In making a representation of a concept of mathematical infinity – that most peculiar of mathematical “numbers” – we proceed to the concept by successive addition and a (reflective) judgment of induction. The pure intuition of space, on the other hand, has the character of a cornucopia; no matter how many subspaces we extract in representation, we can still always extract additional ones. The German word for infinity, *unendlich*, here carries the connotation of “un-end-able.” This logical character of space can be described as the source of the *data* (givens), i.e. subspaces are extractions from an intuitive substratum which stands as the ground for syntheses of empirical series that can never be completed (ended). Even those subspaces extracted in representation can have additional subspaces extracted, again without end, from them. The character of the pure intuition of space therefore carries in its logical essence both infinitude in extension and indefinitude in division. The logical momentum of Quantity in making concepts to describe space is the universal, and the category of Quantity in its concept is totality.

Piaget’s “practical spaces” presentable to the psychologist-observer are to the perception of the Subject merely specific determinations of sensibility in the *Gestalt* of an empirical intuition. We may recall from our much-earlier discussions the syncretic character of childish perception; we are now better able to understand the source of this syncretic character from the Quantity of the idea we make for ourselves of the pure intuition of space. The synthesis of apprehension and the *Verstandes-Actus* are adjudicated by the process of reflective judgment, and these judgments are unconcerned with objects and governed merely by the principal of formal expedience (*Zweckmäßigkeit*). Neither is the pure intuition of space a capacity for representation that is itself concerned with objects. The pure intuition of space is concerned with neither points, nor lines, nor surfaces; it is non-geometrical insofar as either any projective or any metric geometry is concerned. The *Gestalt* (form) of an empirical intuition is whatever arrangement of sensations serves the pure purpose of Reason as judged by reflective judgment, and this is the foundation of childish syncretism.

> Before the age 7 – 8, syncretism may be said to be bound up with all mental events and with nearly all the judgments that are made. For any two phenomena perceived at the same moment become caught up in a scheme which the mind will not allow to become dissociated, and which will be appealed to whenever a problem arises in connection with either of these two phenomena. Thus, when children of 5 – 6 are asked: "Why do the sun and moon not fall down?" the answer does no more than to invoke the other features appertaining to the sun and the moon, because these features, having been perceived *en bloc* and within the same whole as the feature requiring explanation, seem to the child a sufficient reason for the latter. . . Here are some examples: The sun does not fall down "*because it is hot. The sun stops there.* – How? – *Because it is yellow*" (Leo, age 6) [PIAG11: 229].

Other than for some examples and much repetition of the main points quoted above, this is pretty much all that Kant had to say about the pure intuition of space from the theoretical Standpoint. By now perhaps the reason may be obvious. So far as metaphysics in the theoretical Standpoint is concerned, the pure intuition of space is a primitive capacity of mind (in receptivity), and there is little more that can be had *a priori* merely from what is necessary for the possibility of experience. The objective validity of the idea of the pure intuition of space is practical objective validity, and its *logical* character from the theoretical Standpoint, in terms of the categories of understanding, is {totality, limitation, community, necessity}. In the judgment of concepts of space for understanding the logical momenta are {universal, infinite, disjunctive, apodictic}. Metaphysically this is all we can say *a priori* about the pure intuition of space from the theoretical Standpoint. So far as a science of mental physics is concerned, the rest must come from empirical facts of experience and its metaphysic must be based upon evaluation of its characteristics from the judicial Standpoint.

§ 5. Space and Reflective Judgment

We return now to the judicial Standpoint to examine the relationship between the pure intuition of space and the process of reflective judgment. We have earlier seen Kant describe the pure intuition of space in terms of synthetic processes. In the *Opus Postumum* he further tells us

> Without laws no experience can take place and, without a principle of the combination of the manifold in *a priori* intuition, no law. For Knowledge [*Wissen*] exceeds judgment and only makes this capable of thorough-going determination; receptivity discovers certainty in synthetic *a priori* judgments only if the objects of intuition first qualify for this, merely as appearance in my consciousness of myself. For this makes the formal which, pure of everything empirical merely in understanding, assembles rather than sets out a manifold of intuition inasmuch as it emerges from

> the subject's activity. Hence space is not a thing or object [*Sache*][2], and places in it, as points, cannot be aggregated; they all coalesce into one point [KANT10: 175 (22: 36-37)].

Space is not to be regarded as some sort of ready-made template. The manifold in intuition is "assembled" and "emerges from the subject's activity." Kant here makes a rather Piaget-like statement of the practical Nature of the pure intuition of space. The comment that representations in space "coalesce into one point" is also interesting. Since we know Kant's space is not a geometry, this "one point" appears to be a metaphor, much as when we say to someone "get to the point." An empirical intuition is a singular representation and, in terms of Rational Physics, is regarded as the determination of an empirical axiom (the principle of Axioms of Intuition, Chapter 6).

> All points are mathematical; they are not parts but rather determinations [KANT19: 30 (28: 208)].

Because the ground of objective validity for the pure intuition of space lies with activity, and because the matter of composition in an intuition is taken into the manifold as a made-necessary connection of heterogeneous elements, the "assembly process" description of synthesis in the pure intuition of space must supply what is homogeneous in the manifold of an intuition. This can clearly not be ascribed to sensations, which are quite heterogeneous, and so it is some property in the perception of sensorimotor activity that must provide the homogeneity in the coalescence of sensational matters in a singular determination.

In Chapter 16 we looked at the connection between the process of reflective judgment and the adaptive *psyche*. We found this connection in the synthesis in continuity: {objectivity, aesthetic Idea, judicial Idea, Meaning}. We must now seek the connection between sensibility and actions. Here we must proceed cautiously. In Chapter 16 we noted James' model in which sensational data was joined to actions as "impulses" of instincts. Empirically we have seen in this Chapter that the earliest sensorimotor behaviors appear to be organized as scheme-groups. However, in explaining what mathematicians mean by the word "group," it was noted in our analogy that the "actions" (permutations) were "color-blind." If we use James' model of instincts, i.e. if we posit a direct connection of sensation to action response unmediated by any intervening process, we lose this "color-blind" character of our earlier analogy. To put things in terms of the earlier analogy, the "color" of sensibility in intuition is not in the role of a determining factor of motoregulatory action, although this consideration does not necessarily preclude it from occupying the place of a determinable. After all, in our analogy we had 6 basic "actions" in *G*, and the example said nothing about *selection* of an action nor about *ordering* actions in

[2] i.e., space is not a *Sache*-thing.

concatenation.

In James' model we find neither selection nor ordering but rather automatism insofar as first instinctive reactions to stimuli in "first experiences" are concerned. James' two principles discussed in Chapter 16 would seem to rely primarily on maturation in terms of when instincts "ripen" or "fade." If so, this is but a more complex form of automatism. Furthermore, James' picture of instinctive reactions in terms of neurological pathways between sensory cortices and cerebellar or motor cortices would seem to imply a kind of biologically innate structure as a predetermined "given"; such a picture seems to be at odds with the metaphysical picture of the pure intuitions as capabilities and *processes* amenable to accommodation. (As we will discuss later, Damasio's model is not so apparently rigid). Bergson's characterization of instincts, i.e. "instinct implies the knowledge of a matter," is at least less rigid, albeit in Bergson's work one does not find enough discussion of the topic to pin him down to a less ambiguous statement of his position. To reconcile a pure form of intuition (requiring theoretical description in terms of processes) with a connection to motoregulatory capacities, and to do so in such a way that we remain within the scope of the *practical* objective validity of the pure intuitions of space and time, we require something mediate between receptivity in sensibility and action through motoregulatory expression. If sensibility is a determinable and action is a determination, we need an *act* to stand as determining factor.[3]

Because the synthesis of sensibility gives no judgments but merely the matter for judgments, and because the process of determining judgment judges only concepts, an act of judgment involving the connection of sensibility and action can proceed only via the process of reflective judgment. However, the appetitive power – which is the capacity for making objects actual from their mere representation – belongs to pure practical Reason. Because of this, reflective judgment can only present a connection of sensibility and action in terms of a manifold of Desires expedient for the pure purpose of practical Reason. The question thus becomes: What has pure intuition, seen as a capacity for the assembly of intuitions, to do with the possibility of a manifold of Desires? Whatever answer we find can have only practical objective validity, and it must fall under the transcendental judicial principle of formal *Zweckmäßigkeit*.

§ 5.1 The Binding Code Hypothesis I: Convergence Zone Assembly in Perception

Kant did not draw a picture of the pure intuitions as having a mediating connection to action through the process of reflective judgment although his *Anthropology* seems to hint at it indirectly in a few places. He comments [AK7: 167] that the power of imagination is a capacity for intuition

[3] see Chapter 15, §6 on the distinction between act and action.

and that the pure intuitions of space and time go with the capacity for productive imagination in its presentations. Taken in isolation, comments such as this tend to promote the supposition that the capacity for pure intuitions in sensibility involves only the innermost loop in our cycle of thought model (figure 9.3.1). However, this supposition ignores the role of the process of reflective judgment as the process of judgment that marks a mere representation in sensibility as an intuition at a moment in time. Elsewhere [AK7: 162-167] Kant remarks that sensuous feelings are affected in their degree by such things as contrast, novelty, change, and climax. His discussions on this point, although non-technical and merely descriptive, set it in a context in which it is clear that affective perceptions affect consciousness, attention, and objective perceptions. Kant also comes close to, but stops short of, connecting the pure intuition of space in a reciprocal interplay with the process of teleological reflective judgment in the third Critique [KANT5c: 236-239 (5: 364-366)].

In the *Opus Postumum* we find some vague references to space and time, as intuitions, being combined with corporeal movements, these being "together the conditions of the sense-object"; this combination is even referred to as being the cause of movement [KANT10: 163 (22: 442)]. Kant makes various obscure comments to the effect that pure intuition in combination with corporeal movements can be regarded in terms of a "moving power" or a "life power," and refers on occasion to a "principle of movement from desiration" (the *Begehrung nexus* of Desire), e.g. [KANT10: 65 (21: 212)]. The *Opus Postumum* is not remotely close enough to being well organized for us to be able to conclude from it that Kant's system called for an intimate interplay between the pure intuitions as processes and the process of reflective judgment, but from what has been said earlier, this is the interpretation I find to be the most consistent with Kant's system as a whole.

Now, if this philosophical argument has a foundation, then it also has implications for the somatic division of an Organized Being. Specifically, it implies that there should be in somatic representation an interplay between the brain's motor faculties and its sensory faculties, both of which have reasonably well identified gross cortical structures in the cerebrum. No merely empirical evidence of such an interplay can *prove* this theory, but if such evidence can be found it would be at least consistent with it, and its utter absence would speak against the theory. Has any such evidence, either pro or con, been found? The answer here is yes, and this evidence leans in favor of the theory presented here. This evidence is twofold. On the one hand, there are empirical findings pointing to a particular structural schema in brain function. On the other hand, there are theoretical and mathematical arguments pointing to the *requirement* that the brain possess some such structural organization as this. In neither case is the evidence overwhelming, and in both cases there are a large number of details still missing. However, these taken together do help to provide us with a better "picture" of the nature of a relationship between the pure intuitions and reflective judgment. We will present the empirical findings in this section; in the next section we will present the speculative argument.

Spatio-Temporal Integration in the Brain

Although we are far from having anything like a complete description of the brain and its neural organization, neuroscience has amassed a great many findings and the past two decades in particular have seen great strides in our understanding of brain organization. It has long been held that signals produced by neurons carry the information that constitutes the somatic representation of our cognitive knowledge. It is now known, in addition, that this somatic form of representation is physically realized in brain structure by an enormous number of relatively small neural "circuits" that specialize in responding to specific stimuli that correlate to specific features in objects (as these features are viewed in the Piagetian sense of the word 'object'). What is the character of this form of somatic representation? Damasio writes:

> Current knowledge from neuroanatomy and neurophysiology of the primate nervous system indicates unequivocally that any entity or event that we normally perceive through multiple sensory modalities must engage geographically separate sensory modality structures of the central nervous system. Since virtually every conceivable perception of an entity or event also calls for a motor interaction on the part of the perceiver and must include the concomitant perception of the perceiver's somatic state, it is obvious that perception of external reality and the attempt to record it are a multiple-site neurophysiological affair. This notion is reinforced by the discovery, over the past decade, of a multiplicity of subsidiary functional regions that show some relative dedication not just to a global sensory modality or motor performance but also to featural and dimensional aspects of stimuli . . . The evidence from psychological studies in humans is equally compelling in suggesting featural fragmentation of perceptual processes . . . The physical structure of an entity (external, such as an object, or internal, such as a specific somatic state) must be recorded in terms of separate constituent ingredients, each of which is a result of secondary mappings at a lower physical scale. And the fragmentation that obtains for concrete entities is even more marked for abstract entities and events, considering that abstract entities correspond to criterion-governed conjunctions of dimensions and features present in concrete entities, and that events are an interplay of entities.
>
> The experience of reality, however, both in ongoing perception as well as in recall, is not parcellated at all. The normal experience we have of entities and events is coherent and "in-register", both spatially and temporally. Features are bound in entities, and entities are bound in events. How the brain achieves such a remarkable integration starting with the fragments that it has to work with is a critical question. I call it the *binding problem*. . . The brain must have devices capable of promoting the integration of fragmentary components of neural activity, in some sort of ensemble pattern that matches the structures of entities, events, and relationships thereof.[4]

[4] A.R. Damasio, "Time-locked multiregional retroactivation: A systems-level proposal for the neural substrates of recall and recognition," *Cognition*, 33 (1989), pp. 25-62.

The binding problem to which Dr. Damasio refers is widely recognized as one of the key scientific questions to which a solution is being sought by neuroscience. It recalls somewhat the earlier argument by Poincaré in terms of various sensorimotor "spaces," but our present-day knowledge is far more detailed than anything available to Poincaré. We do not perceive the pieces ("fragments") represented by specific neural assemblies in the brain. What we perceive in the phenomenon of mind is the "whole." The binding problem is: Given that fragments are represented in the brain by signals from many neural assemblies, geographically separated in the brain, how are these diverse signals integrated spatially and temporally to produce in perception the cognition of objects ("entities" and "events")?

This problem has been recognized by neuroscientists for many years. For a long time the dominant hypothesis was one inspired more or less by analogies based on forms of logical arguments, exemplified by the form of deductions in symbolic logic, and on an information-processing metaphor more or less based on logical positivism. It is this "traditional solution" that Damasio challenged in the article quoted above on the ground that this model made incorrect predictions of what the effects of particular types of brain injuries from surgery or disease should be. Damasio first describes this "traditional solution" that he attacks:

> The solution, implicitly or overtly, has been, for decades, that the components provided by different sensory portals are projected together in so-called multimodal cortices in which, presumably, a representation of integrated reality is achieved. According to this intuitively reasonable view, perception operates on the basis of a unidirectional cascade of processors, which provides, step by step, a refinement of the extraction of signals, first in unimodal streams and later in a sort of multimedia and multitrack apparatus where integration occurs. The general direction is caudo-rostral[5], and the integrative cortices are presumed to be in the anterior temporal and anterior frontal regions. . . On the face of it, anatomical projections do radiate from primary sensory cortices and do create multiple-stage sequences toward structures in the hippocampus and prefrontal cortices . . . Moreover, without a doubt, single-cell neurophysiology does suggest that, the farther away neurons are from the primary sensory cortices, the more they have progressively larger receptive fields and less unimodal responsivity . . . Until recently, the exception to this dominant view of anterior cerebral structures as the culmination of the processing cascade was to be found in Crick's (1984) hypothesis for a neural mechanism underlying attention.
>
> The purpose of this text is to question the validity of the conventional solution. I doubt that there is a unimodal cascade. I also question the information-processing metaphor implicit in the solution, that is, the notion that finer representations emerge by progressive extraction of features, and that they flow caudo-rostrally. Specifically, we believe that by using this view of brain organization and function the experimental neuropsychological findings in patients with agnosia and amnesia become unmanageably paradoxical. I also suggest that there is a lack of neuroanatomical support for some requirements of the traditional view, and that there are neuroanatomical findings to support an alternative model. Finally, I believe that available neurophysiological data can be interpreted to support the alternative theory I propose.[6]

[5] "Back-to-front," i.e. from the back of the head toward the front of the head. The main sensory cortices in the human brain are located to the rear, while "higher" cognitive functions were thought to be located in the "frontal lobes" located in the front of the head.
[6] *ibid.*

Convergence Zones

Damasio goes on to review the numerous predictions made by the "traditional hypothesis" and various findings, based on patients who have suffered brain damage of various kinds, that contradict these predictions. His specifics include patients with anterior temporal cortex damage, anterior frontal lobe damage, and damage in single-modality cortices. He then cites additional evidence from neuroanatomical and neurophysiological studies. These studies were made possible by the tremendous advances in technology that had occurred since the time the traditional hypothesis had been put forward. Absent of the capacity to carry out detailed studies of the type Damasio was able to call upon, the traditional model described above was really little more than informed guess-work on the part of earlier neuroscientists. With the new experimental evidence available, Damasio was able to build a strong case against the traditional hypothesis. In its place, this same evidence implicated a very different organizational structure for neural signal processing. His solution has eight principal aspects, from and among which came Damasio's hypothesis of convergence zone assemblies (which we have mentioned several times already in this treatise). We will forego a detailed re-accounting of Damasio's technical discussion in this treatise; the interested reader can look up these details for himself in Damasio's paper cited above and in a second paper he published in the *Neural Computation* journal.[7] In summary, the main points of the Damasio model that concern us here are the following.

First, instead of a unidirectional flow of information within the brain, the neural signals flow in both directions, and the "feedback" of neural signals from "downstream" neural assemblies (convergence zones) coordinates the firing of neural assemblies in the "upstream" (more caudal) regions of the brain. It is this feedback mechanism that, in the Damasio model, actualizes the integration of sensory *and motor* activities in the brain. This integration works by "co-activation" of different neural assemblies, i.e. by a cooperative signaling mechanism such that, "The conscious experience of those co-activations depends on their simultaneous, but temporary, enhancement (here called co-attention), against the background activity on which other activations are being played back."[8]

Second, the representations of the physical structure components of entities are "recorded" in the same neural ensembles activated during direct perception, but recall of these representations involves combinatorial arrangements of firing patterns from the neural assemblies involved in

[7] A.R. Damasio, "The brain binds entities and events by multiregional activation from convergence zones," *Neural Computation*, vol. 1, 1989, pp. 123-132.
[8] A.R. Damasio, *op cit.*, *Cognition*.

these ensembles. The "storage" of the information necessary to co-activate the re-presentation of these signaling activities takes place in separate neural assemblies, and these are the convergence zone assemblies. The signaling activities of convergence zone assemblies are called *binding codes*. Convergence zone firing re-activates firing in the "feature-fragment-representing" neural assemblies "upstream" of the convergence zone, and these "upstream" assemblies project their signals to the convergence zone or zones by which they are re-activated during recall.

Third, feature-representing "fragments" are topologically organized in the brain and their firing activities become "associated" (by means of the formation of convergence zones) in the formation of experience. In other words, convergence zones *produce a topological structure* in neural representation. Convergence zone assemblies are located throughout the telencephalon, in association cortices, in limbic cortices, subcortical limbic nuclei, and non-limbic subcortical nuclei such as the basal ganglia. A key thing to note here is that many of these brain structures are heavily implicated in *emotional* response behaviors, i.e. that convergence zone assemblies are not exclusively tied to cognitive perception but also involve to a significant degree those regions of the brain responsible for what we in this treatise have been calling affective perceptions.

Fourth, the geographic location of convergence zones does vary among individuals, but this distribution is not random. There is a strong degree of dependence in their formation and location upon the individual's experience.

Finally, the co-occurrence of activities at multiple sites is achieved by iteration across time phases, that is, intervals in *objective* time as seen by the scientist/observer. Damasio identifies two distinct types of binding code patterns. Type I binding codes inscribe temporal coincidences and aim at reproducing them. The convergence zones that produce them are located in both low and high order sensory association cortices, and they are assisted in the formation of representations of entities by activity in the hippocampal system. (The hippocampal system is heavily implicated in the formation of new memories). Type II binding codes are fired in sequences and produce closely ordered activations in the target cortices. They inscribe temporal sequences (i.e. events), and their convergence zones are the hallmark of motor-related cortices. Their formation and operation is assisted by learning in the basal ganglia and the cerebellum (both of which are heavily implicated in body locomotion).

In short, Damasio's hypothesis is that convergence zone assemblies are responsible for the formation of somatic representations of objects ("entities and events") by means of firing patterns called binding codes. Convergence zone assemblies are also implicated in brain structures known to be involved in emotional and in motor response behaviors, and convergence zones of different types interact and overlap in their signaling. So far as cognitive representations (particularly those we here call intuitions) are concerned, binding code patterns appear to produce a topological organization of somatic representations. The implication that these convergence zones and their binding codes have the same functional significance as the mental structuring function of Kant's pure intuition of space is reasonably obvious. So far as temporal sequences (representations of events) are concerned, the convergence zones and their binding codes involved in this process bear a reasonably obvious functional similarity to the mental structuring function of Kant's pure intuition of subjective time. If Damasio's hypothesis and his convergence zone architecture model stand up in continued experimental studies, it could provide us with something Kant could probably not have dreamed of having, namely a physiological phenomenon that could be tied rather directly (in accordance with the principle of emergent properties in the thorough-going reciprocity between *soma* and *nous*) to our understanding of the pure forms of intuition in sensibility and the workings of the processes of the power of imagination. Whether or not this will be the case must await the outcome of scientific investigations yet to be carried out.

§ 5.2 The Binding Code Hypothesis II: Dynamic Link Neural Networks

Damasio's model arose as a consequence of neurological findings in brain-damaged patients that contradicted the traditional model of hierarchal representation through unidirectional information flow. However, even before these findings and the Damasio model came to light, some theorists working in the field of computational neuroscience had already begun to question the traditional model.

Computational neuroscience can be regarded as the quantitative mathematical theory arm of neuroscience. The adjective "computational" is attached to this part of neuroscience research because of the need to use computers to obtain quantitative answers from the mathematical models proposed by these neuroscientists. One important sub-discipline within computational neuroscience is that which is known as neural network theory, and it was within this sub-discipline that the questioning of the traditional model first arose in response to certain stubborn mathematical problems that had arisen in the early days of neural network theory and had persistently resisted solution. One of the earliest and most important examples of this was recognized as early as 1961 by Frank Rosenblatt, one of the pioneers of neural network theory. This problem was, and is, known as the "superposition catastrophe."

Mathematical neural networks with unidirectional information flow have the ability to classify input signals presented to them. Such networks are said to perform "object recognition" in the sense that activation of a particular "output neuron" is taken to indicate that the presented input pattern is classified by the network as some particular entity or condition. Neural networks with unidirectional information flow are usually called "feedforward networks" in the language of neural network theory.

For example, a neural network can be "trained" to "recognize" characters of the alphabet, or particular spoken words, or to implement logic expressions from propositional logic. Such a network that contains 52 output neurons can be made to correctly classify a two-dimensional arrangement of pixels such that it can "recognize" the 26 letters of the English alphabet and distinguish between upper and lower case letters. However, the word "recognize" as used by neural network theorists is somewhat misleading. The network performs no cognitive function in the sense in which we use the term "cognition" in this treatise. In the mathematical language of neural network theory, such a network is said to "partition" the "input space" (an abstract "vector space" regarded as in some manner characterizing the set of possible input signals). Activation of a particular output neuron (a "grandmother cell") merely indicates that the input signal "belongs" to a particular "partition" or subset of this input space.

Rosenblatt's problem was stated as follows. Suppose that such a neural network was set up to classify patterns in a two-dimensional input grid (a "retina"). Suppose further that the network is set up to "recognize" two "entities" – say, a triangle and a square. Further suppose that the network is also set up to "recognize" when an "entity" is located in either the upper (top) part of the input grid or the lower (bottom) part of the input grid. By looking at which output-layer neurons are activated upon presentation of an input pattern this network can respond correctly to the presentation of a single "object" – e.g. "triangle in the upper part" or "square in the lower part." The problem arises when the network is presented with two "entities" – e.g. an upper triangle and a lower square. The output activations would then indicate triangle, square, upper, and lower; however, we could not tell if this meant "triangle in the upper part and square in the lower part" or if it meant "triangle in the lower part and square in the upper part." When two sets of output neurons, each set representing membership in one particular classification, are simultaneously activated, the information that discriminates their membership in the original sets is irretrievably and automatically lost. This is the "superposition catastrophe."

It can be, and sometimes is, argued that Rosenblatt's problem is unfairly stated. By adding more output discriminations, multi-object input cases can indeed be properly classified and the superposition catastrophe thereby avoided. This approach is sometimes called "grandmother cell" encoding. In principle this is true, but there is a problem with this approach. Suppose a network is required that classifies all binary logic functions of n binary inputs. The number of possible functions for this situation is 2^{2^n}, a number that quickly becomes extremely large as n increases.

By the time *n* = 6, the number of "grandmother cell" output neurons required to uniquely represent all the possible functions is about 100 *million* times greater than the total number of neurons in the human brain! In the *generalized* Rosenblatt "retina problem" we either avoid the superposition catastrophe at the expense of a "combinatorial catastrophe" or else we must impose *a priori* limits on the number of classifications that the network is to be able to implement.[1]

There is another problem that attends the "grandmother cell encoding" scheme in feedforward neural networks. In biological neural networks, neurons die. The death of a "grandmother cell" (or the death of a sufficient number of cells in the "chain" leading to the grandmother cell) effectively wipes out the "cognition" that activation of that cell is supposed to represent. Hence the grim neurosurgeons' joke, "there go the piano lessons." Moshe Abeles, among others, has made theoretical calculations which show, at cell death rates that are fairly typical in adult human beings, if the brain were organized along the lines of simple feedforward chains of neurons leading to a final grandmother cell we would all be functionally brain dead within about 5 years. This untimely end is shown to be avoidable if neural network structure is organized into redundant, parallel-interconnected cell groups dubbed "synfire chains,"[2] and this type of structure is more or less compatible with the structures normally used by theorists in feedforward neural networks. However, there is again an issue in the mathematics.

In 1969 two researchers, Marvin Minsky and Seymour Papert, published a book that made them pariahs in the world of neural network research. In this book, Minsky and Papert presented a number of mathematical theorems that demonstrated serious and fundamental limitations for what the feedforward neural networks of the 1960s could achieve.[3] Although advances in neural network theory made in the late 1970s and early 1980s answered some of the problems Minsky and Papert presented, it is by no means the case that *all* the mathematical issues they raised have been taken care of. Some of the most serious of these issues are still lacking a satisfactory answer. One of these is called the "stratification problem." A proper statement of this problem involves advanced concepts in mathematics, but in more accessible language the issue is this: As the size of the input space increases, the accuracy with which each neuron in a feedforward network must carry out its "calculations" increases at a geometric rate. If this required level of accuracy is not achieved the network quickly becomes unreliable, i.e. cannot be made to properly carry out its function. The size of the "input space" faced by the central nervous system is such that unidirectional information flow processing would certainly fall victim to the stratification

[1] The original Rosenblatt problem described above does not require as many grandmother cells as the binary function problem but more general "retina problems" exist that require enormously *more* than this.
[2] M. Abeles, *Corticonics*, Cambridge, UK: The Cambridge University Press, 1991.
[3] M. Minsky and S.A. Papert, *Perceptrons*, expanded ed., Cambridge, MA: The MIT Press, 1988.

problem even if these networks are structured as feedforward-only synfire chains.[4]

It has been known from the earliest days of neural network theory that the "calculation accuracy" of biological neurons falls far short of what the traditional unidirectional flow model appears to mathematically require.[5] There seems to be only one answer to these issues, and this is: to abandon the traditional model. One of the earliest theoreticians to take this step was Christoph von der Malsburg, who published in 1981 a new proposal for how to model neural processing.[6] Malsburg's approach is today known as the correlation theory of brain function. More recently, Malsburg published a review article in which can be found a general overview of the issues and problems attending the traditional model, and how the correlation theory of brain function proposes to be a solution to the binding problem.[7]

The correlation theory of brain function and Damasio's model were developed independently of one another, and at the time of this writing the two models have not really been put together in a common framework. However, Malsburg's theme and Damasio's model are in most respects quite compatible with one another, and it is possible that future developments will show them to be, in a manner of speaking, related as matter (Damasio) to form (Malsburg). Probably no one would be willing at the current time to stake his or her reputation on this outcome, but at the very least the two models share many ideas that are conformable. My opinion is that a synthesis of Damasio's model and Malsburg's mathematical theme is probably the best approach available to us at the present time. If such a unification can be effected, it would constitute a new type of theoretical neural network structure of the sort dubbed a "dynamic link architecture" by Malsburg.[8]

Malsburg envisions neural networks as so-called "topological networks." In a topological network the individual neurons make up the members of the set X (see our earlier definition of a topological space in §2.2). The subsets of T are determined by the interconnections among the neurons. These interconnections do not have fixed interconnection strengths ("synaptic weights"), but rather the effect on a particular neuron of signals coming from the other neurons in the

[4] Cell death is the ultimate case of loss-of-accuracy, but the problem would exist even if cells were immortal.

[5] J. von Neumann (1952), "Probabilistic logics and the synthesis of reliable organisms from unreliable components," in *Papers of John von Neumann on Computing and Computer Theory*, W. Asprey and A. Burks (eds.), Cambridge, MA: The MIT Press, 1987.

[6] C. von der Malsburg, "The correlation theory of brain function," in *Models of Neural Networks II: Temporal Aspects of Coding and Information Processing in Biological Systems*, E. Domany, J.L. van Hemmen, and K. Schulten (eds.), NY: Springer-Verlag, 1994, pp. 95-119.

[7] C. von der Malsburg, "The what and why of binding: The modeler's perspective," *Neuron*, vol. 24, pp. 95-104, 1999.

[8] C. von der Malsburg, "Dynamic link architecture," in *The Handbook of Brain Theory and Neural Networks*, second ed., M. Arbib (ed.), Cambridge, MA: The MIT Press, 2003, pp. 365-368.

network causes the strength of the synaptic interconnections to be modulated. This constitutes the "dynamic link" property of the network. It is by means of these dynamic links that different subsets of T can be established. Provided that the conditions of the definition of a topological space are met with, such a structure would instantiate a topological space.

Although at the present time it is merely conjecture whether or not dynamically-interconnected neurons actually form topological spaces in the brain, there is relatively little room to doubt that biological neurons are interconnected by modifiable synaptic weights, and there is likewise little room to doubt that, at least in the neocortex, collections of neurons do form themselves into subsets of cell groups. These cell groups are usually called by such names as "functional columns", "slabs", and "barrels." There are experimental means of testing for the existence of functional columns in the neocortex. Some of these tests have turned up evidence having the interesting implication that, at least in the visual cortex, there are more functional columns present than there is room to hold them *unless the functional columns are dynamically linked*, i.e. unless they can reconfigure themselves to re-use the available neurons in forming different functional columns.[9] Thus, the evidence at this time is not inconsistent with Malsburg's general idea of a dynamic link architecture model of topological networks.

Moreover, it is known for a fact that neuronal interconnections are modified by experience. This is particularly the case in the developing brains of young children, where changes in cortical structures are strikingly visible. Although as yet computational neuroscience is far from having a unified theory for explaining the observed phenomena of neural development and experience-mediated neural network plasticity, the past two decades have seen the emergence of wholly-new ways of thinking about neural network modeling[10] and synaptic plasticity.[11] Thus, for reasons both theoretical and neurological, it has become clear that the days of the traditional model have probably come to an end insofar as theoretical neuroscience is concerned.

§ 5.3 The Interactive Roles of Pure Intuition and Reflective Judgment

The epistemological, empirical, and mathematical considerations just reviewed put us in a position where we can now elucidate the interactive roles of Kant's pure intuitions and the process of reflective judgment. As we discuss this *theoretical* model, we will always bear in mind

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[9] see E.L. White, *Cortical Circuits*, Boston, MA: Birkhäuser, 1989.
[10] see R. Eckhorn, H.J. Reitboeck, M. Arndt, P. Dicke, "Feature linking via synchronization among distributed assemblies: Simulations of results from cat visual cortex," *Neural Computation* 2, 293-307 (1990).
[11] see E.L. Bienenstock, L.N. Cooper, and P.W. Munro, "Theory for the development of neuron selectivity: Orientation specificity and binocular interaction in visual cortex," *Journal of Neuroscience*, vol. 2, no. 1, pp. 32-48, Jan., 1982.

that, so far as a theory of mental physics is concerned, the basis for the objective validity of this model is rooted in the judicial Standpoint of the Critical Philosophy. Our interpretations given here are set in and moderated by the requirements of *practical* objective validity.

Our first consideration is given to the functional nature of Kantian space. Space is a structuring function of sensibility that provides the form of sensible representation for objects regarded as Damasian "entities." The formative capacity of the pure intuition of space is not a geometry but it can be viewed as **a topological synthesis** in general. The pure intuition of space from the theoretical Standpoint is the capacity to organize sensations in sets of subsets constituting a topology. This is not to say that sets within the topology T are pre-fixed or innate. Practical objective validity for the pure intuition of space requires that we regard space not as a predefined topology but **as the capacity to synthesize topological assimilations** of the data of sensation. The representation in *soma* of this assimilatory action subsists in some binding code neuronal signaling pattern, perhaps Damasio's "type I" binding code. A topological space made possible by the synthesis of the pure intuition of space can be called an outer-objective topology.

Although we have not yet discussed Kant's pure intuition of time at the same level of detail given to Kantian space in this chapter, we will later see that the pure intuition of time has a similar interpretation, but one applied to order structure rather than topology. A helpful analogy is to think of the pure intuition of space in terms of the specific permutations in our group example, but to think of the pure intuition of time as the synthesis of a sequence of concatenations. In other words, space goes to the possibility of the elements of *G* in our example, but time goes to the concatenation operation "$\bullet$" in that example. In his analysis of Kant's system, Paton remarked,

> Above all we must recognize that no sufficient account of space and time can be given if we isolate them from one another. For my own part, though I speak as the merest amateur in these matters, I do not see why it should be impossible to re-state Kant's assumptions so as to fit in with the modern physical doctrine. It seems to me at least possible that space-time is the condition or form of all appearances given to sense; that we can gradually sort it out, by a special kind of abstraction, from the appearance of which it is the form, and study it mathematically as an individual whole which is intelligible through and through; and that in doing so we can discover laws to which the world of appearances must conform [PAT1: 163].

Paton is correct that the pure intuitions of space and time cannot be isolated from each other insofar as our theoretical understanding of empirical intuitions are concerned. His "re-statement of Kant's assumptions" we can better understand not as re-statements of principles from the theoretical Standpoint but, rather, as an explanation of the *function* of the pure intuitions of space and time based on understanding that the objective validity of Kantian space and time is practical objective validity to be viewed from the judicial Standpoint. What we are engaged in at this point in the treatise is the making of intelligible ideas of Kantian space and time by which we can view them *as synthesis operations* from the theoretical Standpoint. The representation in *soma* of the assimilatory acts of the pure intuition of subjective time subsists in neuronal binding code signals for "events," perhaps binding codes of Damasio's "type II" class.

Note carefully that I say these pure intuitions *subsist in* binding code representations in *soma*, not that binding codes *are* representations of the pure form of intuition. We cannot obtain scientific experience of binding codes other than empirically, and this means we must form empirical intuitions and concepts in the process of understanding these objects of appearance. The capacity for pure intuition is a synthetic capacity, not the representation forthcoming as a consequence of the exercise of this capacity. The pure forms of intuition, viewed as supersensible objects of an idea, are *Unsache*-things – "happenings." All our theoretical ideas of the characteristics of the pure form of intuition are ideas of its accidental Nature (*Existenz*).

The theoretical character of the pure intuition of space contains no concept of a judgmental or decision-making capability. Space belongs to the power of receptivity but must be viewed as a determinable capacity. An empirical intuition is a determination. This parallels what we said earlier in this treatise concerning the categories and the process of determining judgment. Determining judgment is the *a priori* capacity to employ the categories as rules for the determination of concepts (and concepts are rules for the re-presentation of intuitions). But determining judgment does not have the power to determine its own employment and, rather, is under the supervision of the power of speculative Reason (ratio-expression). In a similar vein, the synthetic capacity of pure intuition does not determine its own employment but, rather, is adjudicated by another process, namely that of reflective judgment.

The principle of all reflective judgments is the principle of the formal expedience of Nature, and reflective judgment is tasked with making a system of Nature. Affective perceptions are, in this context, reflections of coherence within a system of Nature, and the expedience of the representation of an empirical intuition is judged on this basis. The judgment of this expedience also implicates a relationship to motor spontaneity because maintenance of an inexpedient state within the faculty of pure consciousness is not tolerated by practical Reason. An Organized Being's motor capacity is one of its means for effecting changes in its state-of-being, thus an act of reflective judgment determining an empirical intuition at a moment in time is also *in commercium* a determination of possible motor actions according to the same principle of formal expedience. As noted back in Chapter 9, "a *commercium* is thus needed for *composito substantiali*. The form of *compositi substantialis* thus rests on the *commercio*" [KANT19: 19-20 (28: 195-196)]. The vehicle of this *commercium* is the representation of possible appetites, and here is where our previous discussion in Chapter 12 of appetite, desire, and desiration obtain their relationship to objective cognition. Reflective judgment presents possible appetites (as a manifold of Desires) to practical Reason, and the state of formal expedience for a system of Nature in the synthesis of apprehension via the pure intuition of space is a contributing factor in the determination of possible appetites. The synthesis of the pure intuition of space in sensibility stands in a Relation of community with motoregulatory expression, and the determination of this reciprocity falls to the process of reflective judgment at the service of pure Reason.

This relationship is illustrated in Figure 17.5.1 below. The figure is a generalized extension of the cycle of thought model illustrated earlier in Figure 9.3.1, and places the capacity for moto-

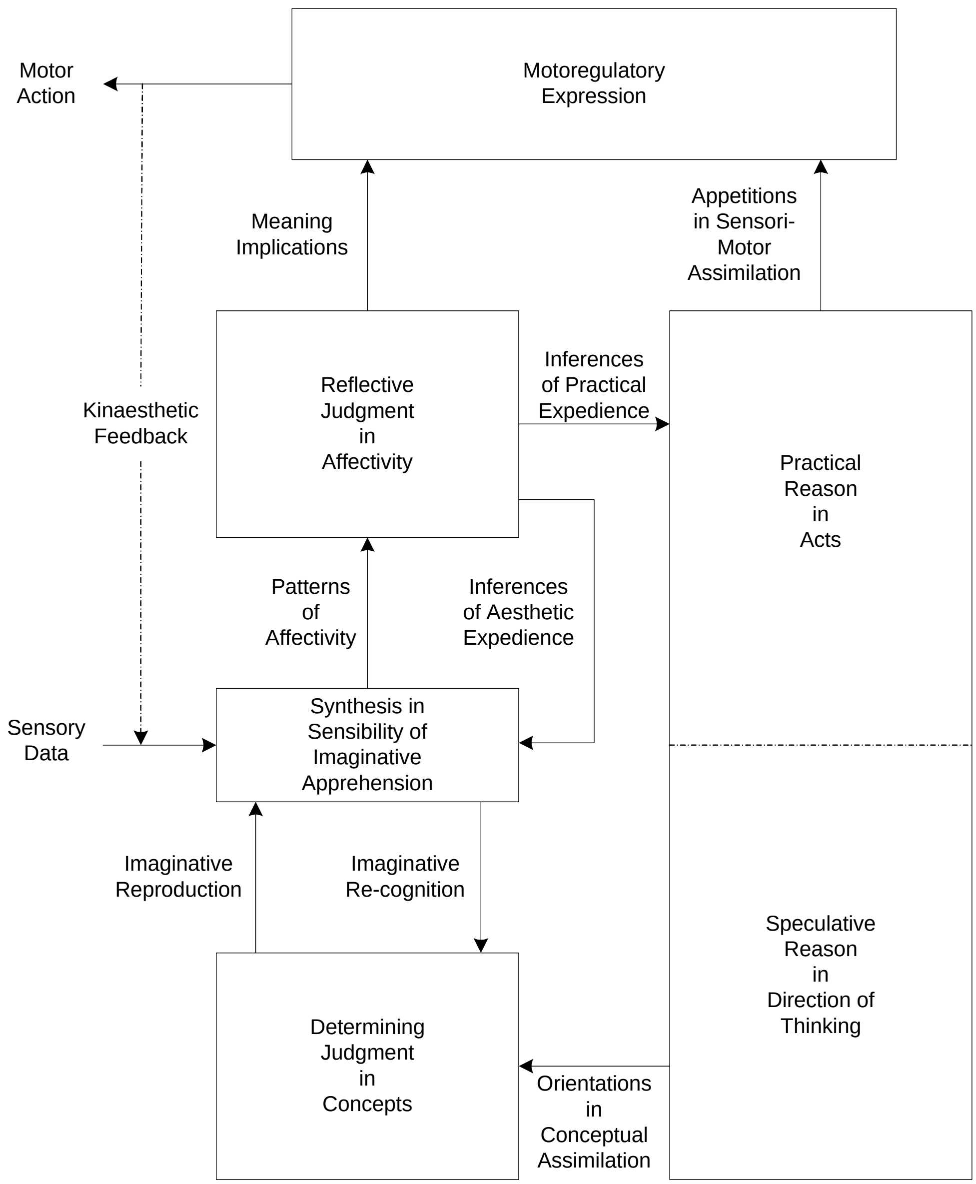

Figure 17.5.1: Organization of information flow in judgmentation and reasoning

regulatory expression in its proper relationship to the conscious mental processes of receptivity, judgment, and Reason. From the judicial Standpoint the process of reflective judgment makes judgments of three types. Inferences of aesthetic expedience affect the synthesis in apprehension/comprehension in sensibility and, in relationship to the pure synthesis of space in intuitive representation, these inferences exercise a role as a kind of veto power over the topological *Gestalt* of possible representations of intuition. This is based on the formal expedience of these representations. Inferences of practical expedience present possible appetites (Desires) for consideration by pure practical Reason. These inferences are non-cognitive and are tied to possible motoregulatory expressions, which are depicted in the figure as meaning implications (the third type of reflective judgment depicted above). Also, from the judicial Standpoint the regulative directions given by pure speculative Reason to the process of determining judgment are regarded as *orientations* in cognitive assimilation (ratio-expression). Pure practical Reason exercises a **veto power** over the possible motoregulatory expressions judged expedient by reflective judgment, and expressions not so vetoed are the appetitions in sensorimotor assimilation of the determination of the appetitive power of pure Reason. Kinaesthetic feedback in the spontaneity of motoregulatory expression affects sensibility, and this closes a sensorimotor loop of {sensibility – reflective judgment – motoregulatory expression} in which sensibility and motor action are placed in a Relation of community. Figure 17.5.1 outlines the flow of information within the overall process of judgmentation in general (*Beurtheilung*).

§ 6. The Topological Synthesis of Space

We have just given an explanation of the pure form of the intuition of space as being a topological synthesis of sensibility. There are, however, some important points that have to be made regarding how we are to view this synthesis if its objective validity is to be maintained.

§ 6.1 The Topological Object

In formal mathematics, a topological space is defined by a set X and a collection T of subsets, $T_i \subseteq X$, such that T exhibits the properties described earlier. Mathematically, a set is a collection of "points" (in the language of point set topology) and our first issue is: What is X and what do we mean by the "points" in X insofar as the pure intuition of space is concerned? As the form in outer sense of an empirical intuition, the pure intuition of space provides the manifold in sensation. Does this mean we are to regard the "points" as being "sensations"?

The answer to this question must be no. To so regard the points in X is to regard "sensations" as some kind of "atomic" representation (to use James' characterization). We have no objectively valid ground for so viewing "sensations" and could base such a representation only upon an analogy that lacks a sufficient objective ground. From the theoretical Standpoint "sensation" can be regarded as nothing else than the effect of an object on the capacity for representation insofar as we are affected by that object. Were we to posit atomic "sensations" as the points in X, we would find ourselves obligated to try to catalog all the possible "sensations" much as early psychologists attempted to compile a catalog of "emotions." Furthermore, as the matter of an empirical intuition, "sensation" implies consciousness, and, likewise, as the matter of affective perception (i.e. as "feeling"), sensation (feeling) also implies consciousness. Psychology defines sensation in general as "any unelaborated elementary experience of feeling or awareness of conditions within or outside the body produced by the stimulation of some receptor or receptor system, a *sense datum*." The pure *a priori* intuition of space is, transcendentally, a capacity necessary for the possibility of experience, whereas X and the points in X are logically prior to experience because they are logically prior to the synthesis of a topology T. As objects they are merely *intelligible* objects and not appearances nor matter in appearances.

However, if the points in X are not sensations or feelings, what is left for them to be? To obtain an objectively valid *Realerklärung* of this we must employ our applied metaphysic of the data of the senses, and in this we must adopt the judicial Standpoint. From Chapter 6 we recall that the data of the senses was described in terms of the following titles: physiological idea (integration), seeming (subcontrarity), emergent properties (the transitive in Relation), and state of satisfaction-dissatisfaction (the determinable). The judicial Standpoint is the Standpoint from which this idea is examined under a synthesis of the practical viewed as the theoretical, hence a *mathematical* idea of X and its points must be an idea of *functional* consequences of X that we will call "practical accidents" of receptivity. This is an epistemological, not an ontological, idea.

When we regard the Organized Being as an Object in Nature, and in the context of the pure intuition of space, its capacity of receptivity (viewed in terms of Quantity) is a whole of possibilities for the manner in which the Organized Being can be co-determined in outer sense in reciprocity with its environment (passiveness of receptivity). The physiological idea is the idea of a doctrine of the entirety of the functional organization of an Organized Being, and this idea limited by the context of the pure intuition of space in receptivity bespeaks of the accidents of the senses. To appreciate this from the judicial Standpoint, it is helpful to recall Kant's comment that the judicial Standpoint addresses his "What may I hope?" question. Seen in this way, the set X viewed as an aggregate is the sum-total (union) of all possible manners in which sense can affect the Organized Being in an *effective accident* of *Existenz*. From the judicial Standpoint it is the Organized Being as Subject who is the Object considered here. The idea of a set X is an idea of an accident of the Subject's *Existenz*, and by *effective* accident I mean the form of a composition that represents the causality of receptivity (as obscure representation) in sensuous *circumstances*.

> Occurrence is a single act with its result; event regarded in its connection is called the connection or Relation with other things at the same time (the outer connection in which the occurrence happens is circumstance – local circumstances, insofar as they concur with the event, are called temporal opportunities – timeliness). Circumstance, or the relating of occurrence with respect to the simultaneous, is a particular situation. Coordinate co-causes hide in the particular situations. . . The effect testifies to the cause. The action indicates the ground of the causality, i.e. the determination of the reason to take action, through which the *causatum* exists; for the effect shows only living powers, applied powers, not dead powers, which have no effect. . . The *nexus* between an efficient cause and the effect is an effective *nexus* [KANT19: 202-203 (29: 845-846)].

In an effective accident of *Existenz* in sensibility, its character of being part of a circumstance is the homogenous in the Quantity of receptivity, hence we can call the form of the matter of X the **sensuous circumstance** of sense. Judicially considered, sensuous circumstances make up the *materia circa quam* for acts of topological synthesis.

The Quality of seeming is an inducement to judgment. I can hardly be said to have "received" anything in receptivity unless this "reception" can be viewed as a ground for change in my state of being. But change cannot be imputed unless it has as a practical consequence an act of judgment realized in an action (physical, mental, or both). Thus the common coalition of every point in X is the possibility of registering an effect of some degree of *Lust per se*, and this is where we find homogeneity in the matter of the matter of X. Taking this in conjunction with the Quantity of composition, the matter of X is the composition of sensuous circumstances of *Lust per se* that contains a ground for performing an act of judgment.

The phenomenon of attention is the emergent property of sensuous circumstances.

> We have *Lust* or *Unlust* without having to desire or detest, e.g., if we see a beautiful region then it enchants us, but we will not on that account wish at once to possess it. *Lust* or *Unlust* is thus entirely different from appetitive power. But on the other hand we can desire or detest nothing which is not based on *Lust* or *Unlust*. That for which I have no *Lust* I also do not want. Thus *Lust* or *Unlust* precedes desire or detestation. But still I must first know what I desire, likewise for what I have *Lust* or *Unlust*; both are based on the faculty of knowledge. There are also many representations which are combined with neither *Lust* nor *Unlust*, and thus the faculty of knowledge is wholly distinct from the feeling of *Lust* or *Unlust*. These three chief powers of the soul combined make up its life.
>
> Consciousness is the principle of the possibility of understanding, but not of sensibility. Consciousness according to choice is attention – the replay of that is abstraction. The self underlies consciousness and is what is peculiar to spirit. . . Abstraction is the actualization of Attention, whereby only a single representation is made clear and all the remaining are obscured. Attention does not stop with abstraction, but rather it is only directed from one or several Objects to one, and all the remaining representations obscured, and the one clear [KANT19: 247-248 (29: 877-878)].

The Relation of emergent properties in the data of the senses is the idea of the transitive. The transitive is that which is common in the form of several representations that are otherwise distinct and heterogeneous, but which nonetheless are joined with each other in the transitive term. In sensibility the capacity for attentiveness, described by Kant above, provides for such a joining-in-common of what is otherwise diverse in sensuous circumstances.

In the logical divisions of *nous*, the synthesis in sensibility and the processes of judgment are viewed as distinct and heterogeneous active capacities, and the transitive is the idea of that which can be regarded as some factor they all share in common. In the context of a manifold in X, the feeling of *Lust per se* (in affective perception) and sensation in an empirical intuition are presentations that are quite heterogeneous in their logical essence. All the same, the *nexus* of a manifold in presentation is the *a priori* combination of matters for which their combination in the manifold is viewed as necessary despite their heterogeneity. The *practical necessitation* of this combination is based upon the necessity for the presentation of appearances to be possible. This requires representation in intuition, but we have already seen that the possibility of intuition requires an act of reflective judgment, that the matter of reflective judgment is affective perception, and consequently *nexus* in the manifold of X implies connection of affective and objective matters of sense-data (the given-in-sense).

We have long taken regard of this in this treatise; the process of this formation of conscious presentations was first discussed in the discussion of the three-fold synthesis of the *Verstandes-Actus*. Because each point *x* is regarded as a sense-datum in a possible sensuous circumstance (a "particular situation" in Kant's terms quoted above), the gathering of several such points together in a subset T_i of X makes a manifold we can call the **matter of Attention**, and the idea of such a manifold is the idea of a *Gestalt* for sensibility in the faculty of pure consciousness.

This discussion is, of course, a discussion taking place in the context of *nous*. However, it is instructive to note that Damasio's idea of binding codes has reference to the phenomenon of attention in the context of representation in *soma*.

> In a system that produces multiple-site activations incessantly, it is necessary to enhance pertinently linked sites in order to permit binding by salient coincidences. I use the term attention to designate the 'spotlighting' process that generates simultaneous and multiple-site salience and thus permits the emergence of evocations. Consciousness occurs when multiple sites of activation are simultaneously enhanced in keeping or not with real past experiences. . . As defined here, attention depends on numerous factors and mechanisms. First, there is a code for enhancement of activations that is part of the record of the activation pattern it enhances. Type II convergence zones are especially suited to this role. Secondly, the state of the perceiver and the context of the process play important roles in determining the level of activations. The reticular activating system, the reticular complex of the thalamus, and the limbic system mediate such roles under partial control of the cerebral cortex.

> The evocations that constitute experienced recall occur in specified sensory and motor cortices, albeit in parcellated fashion. Experienced recall thus occurs where physical structures of external entities or body states were mapped in feature fragment manner, notwithstanding the fact that a complex neural machinery made up of numerous other areas of cortex and subcortical nuclei cooperates to reconstruct the co-activation patterns and enhance them.[12]

The brain structures mentioned here by Damasio are those which neuroscience regards as neural substrates of both cognitive and affective phenomena. It is a reasonable speculation, but only a conjecture at the present state of scientific knowledge, to posit the neural activity patterns to which Damasio refers as the somatic correlates of the "points" x in X. "Enhancement," i.e. binding producing equilibrium in some of these patterns and the extinguishing of others, can be seen in this context as the neural equivalent of forming a subset T_i within the set of all possibilities of sensuous circumstances X.

Also noteworthy in Damasio's model is the involvement of motor cortices in "spotlighting" attention. We have noted on several prior occasions that one intuition at a moment in subjective time "grows out of" a mathematically prior marked moment. The neural process Damasio describes in regard to Type II binding codes is such a "growth process" described in objective time. Here we have an example of where the theory regards objective and subjective times as not being the same. The latter, of course, is the ground of the possibility of ideas of the former.

We can furthermore note the interesting juxtaposition of Damasio's described mechanism of the attention process with James' psychological theory of volition. James regarded "will" in terms of an "ability" or "effort" put forth to keep particular "impulses" in play. "Willful actions" in the Jamesian sense are "events" in the Damasian sense, and in Damasio's model this especially involves binding codes of Type II.

To sum this up: the possibility of the form of the form of X subsists in the formation of subsets T_i in the context of regarding such subsets as constituting matters of Attention. Those points x excluded from a particular attended-to subset T_i correspond to Kant's idea of obscure representations. The manifold of X is a manifold of sensuous matters of Attention.

Finally, we come to the idea of Modality in representation of X. The idea of Modality in the data of the senses is the idea of the determinable. We do not say that sensuous circumstances in and of themselves determine the acts and actions of the Organized Being; rather, it is the full interplay of receptivity, judgmentation, practical choice, motoregulatory expression, and the *Lust-Kraft* of the adaptive *psyche* that combine to produce the circumstance. Thus, sensuous circumstances are determinable, not determining. This is in keeping with the role of Modality as not adding to our understanding of a particular object. Rather, it is the matter of form, i.e. how we

[12] *op cit.*, A.R. Damasio, *Cognition*.

understand the manner by which a representation coheres with the powers and processes of the determining Subject. X is that in receptivity which is determinable through the full interplay of the noetic, psychic, and somatic *Kraft* of the Organized Being. "What" is determinable through the synthesis of apprehension is the state of satisfaction-dissatisfaction in the data of the senses.

§ 6.2 The Topological Synthesis

Neither the idea of X nor the idea of subsets T_i that we have just discussed constitute, by themselves, a topological space. A topological space requires a topology T, and this set-of-sets-T_i must have particular properties if it is to constitute a topology. The barest properties required for it are those stated earlier in the mathematical definition of topological space, i.e., 1) that the intersect of any two T_i sets is also in the collection T, and 2) the union of any collection of sets in T is also in T. (Here we are to understand "union" and "intersect" in the usual mathematical way). This means that *how the T_i are formulated determines whether or not we have a topological space*. We therefore require rules for this formation of subsets in the context of a collection T of the subsets so formed. Seen in this way, **the pure intuition of space is a faculty of rules for the construction of a topology T**. The pure *a priori* intuition of space is not a pre-defined topology but, rather, the capacity ("know-how") to *build* a systematic topology. It is a topological *synthesis*.

Nothing in the idea of the pure intuition of space requires such a topology to emerge full-grown at the onset of experience. The idea permits limitations *in* space, i.e. "sub-spaces." Nor does the metaphysic of space provide us with positive criteria of what constitutes a spatial (topological) system. What it provides are rather negative criteria, i.e. what can *not* be assimilated into a single, systematic topology of intuitive space. Childish syncretism is one outcome of this. To paraphrase James, everything that *can* be assimilated in one intuition *is* so assimilated.

How are we to understand this process to work? In other words, what is contained in our theoretical conceptualization of how the pure intuition of space carries out its business? Kant does not provide us with a great deal of help in answering this question, but he did provide one instance of reasoning about the conceptualization of an empirical objective space. This came in his 1786 work, *Metaphysical Foundations of Natural Science*.

As usual, Kant starts his metaphysics of natural science by dividing the topic into the four heads of representation (Quantity, Quality, Relation, and Modality). In this instance, these four titles are named phoronomy, dynamics, mechanics, and phenomenology. The metaphysic of "physical space" falls under the title of phoronomy, which considers the composition of motion and is today, bereft of all philosophical considerations, called "kinematics."

The Metaphysic of Motion and Empirical Space

So far as sensible objects are concerned, Kant defines the *matter* of such an object to be that which corresponds to sensation in the intuition of that object's appearance. When further restricted to the topic of phoronomy,

> *Matter* is the *movable* in space. The space which is itself movable is called material, or also *relative space*; that in which all *motion* must ultimately be thought (which is therefore itself utterly immovable) is called pure, or also *absolute space* [KANT15a: 194 (4: 480)].

This explanation starts out in a manner with which Newton would likely have felt completely at ease. However, this situation doesn't last very long in Kant's metaphysic. The warning rattle that Kant is about to do away with Newton's view of "absolute space" is given in the innocent-looking phrase "must ultimately be *thought*." (Leibniz, of course, would likely have already been put on his guard by the appearance of the term "absolute space" since he denied there could be any such thing). We already know, of course, where Kant will go with this "pure absolute space."

Kant next briefly described to what issues the topic of phoronomy is limited.

> Since in phoronomy nothing is to be talked about but motion, the subject of motion, namely matter, has here no other attributes annexed to it than *movability*. It can itself accordingly hold good for one point, and in phoronomy we abstract from all inner properties, hence also from the magnitude of the movable, and have to do only with motion and what in this can be considered as magnitude (speed and direction) [KANT15: 150 (4: 480)].

The comment about phoronomy being valid for consideration of motion at a single point is a lead in to the idea of a "place in space." In physics it is common to describe the motion of bodies in terms of the motion of a single point in objective space, namely a body's center of mass. Kant is allowing here that this practice is valid, and this is what he is talking about when he says that in phoronomy we "abstract from all inner properties" of "the magnitude of the movable" (e.g. the composition in Quantity of a body). The validity of talking about motion in terms of points will be important to us later in more general considerations of the topological synthesis of space.

Now physics takes for its topic the description of what is usually called "external physical nature" – meaning that physics has nothing to do with psychology except perhaps indirectly in attempting to provide a "dead matter" (to use Kant's term) substrate for brain theory. External objective space is an object for physics. But what sort of object? That is what Kant next explains.

> If I am to explain the concept of matter, not through a predicate that belongs to it itself as an object, but only by the relationship to the faculty of knowledge in which the representation can first of all be given to me, then matter is every *object of the outer senses*, and this would be merely the

> metaphysical explanation thereof. Space, however, would be merely the form of all outer sensuous intuition . . . *Matter*, in opposition to *form*, would be that in outer intuition which is an object of sensation, and thus the properly empirical of sensible and outer intuition because it can in no way be given *a priori*. In all experience something must be sensed, and that is the real of sensuous intuition, and therefore the space in which we are to arrange our experience of motion must also be perceptible – that is, it must be signified through what can be sensed, and this, as the embodiment of all objects of experience and itself an Object of experience, is called *empirical space*. But this, as material, is itself movable. But a movable space, if its motion is able to be observed, presupposes in turn an enlarged material space, in which it is movable; this latter presupposes in precisely the same way yet another; and so on *ad infinitum* [KANT15: 150-151 (4: 481)].

Here Kant and Descartes are not all that far apart on agreeing to what must be the real ground of any conceptualization of empirical space. Descartes said we know of space only because we can perceive extension in bodies; Kant is saying that a concept of an empirical space can only be built out of sensible appearances, and that its role is the formal arrangement of sensation by which we embody an object of experience. Note, however, that the conception of such a space takes it ground from the arrangement of the experience of *motion* (and therefore its conception cannot be freed from the participation of the pure intuition of time). Here is where Kant and Descartes decisively part company. For Descartes the perceptibility of space is like a snapshot perception of extension; for Kant empirical space emerges in a process of synthesis in subjective time.

Note, too, that the empirical space thought to contain an object is not a "complete space." It is conceptualized as itself being "movable" and, therefore, this characteristic of empirical space requires as a substratum some kind of "All-of-Space" (just as the reality of an object must be seen as a special limitation in an "All-of-Reality").

> Thus all motion that is an object of experience is merely relative; and the space in which it is observed is a relative space, which itself moves in turn in an enlarged space, perhaps in the opposite direction, so that matter moved with respect to the first can be called at rest in relationship to the second space, and these variations in the concept of motions go forward with the variation of the relative space ad infinitum. To assume as *given in itself* an absolute space denominates something which can be observed neither in itself nor in its consequences (motion in absolute space), that is, one such that it cannot also be an object of experience, because it is not material, for the sake of possible experience - which, however, must always be arranged without it. Absolute space is thus *in itself* nothing and no Object at all, but rather means only any other relative space, which I can think beyond the given space, and which I can only defer to infinity beyond any given space, so as to include it and suppose it to be moved. Since I have the enlarged, although still always material, space only in thought, and since nothing is known to me of the matter that designates it, I abstract from the latter, and it is therefore represented as a pure, non-empirical, and absolute space, with which I compare any empirical space, and in which I can represent the latter as movable so that the enlarged space always counts as immovable. To make this into an actual thing is to denominate the *logical universality* of any space with which I can compare any empirical space, as included therein, into a *physical universality* of actual compass, and to misunderstand reason in its Idea [KANT15a: 195 (4: 481-482)].

Here we find a statement of one of Einstein's central tenets nearly a century before Einstein was born. The objective validity of absolute space (i.e. the pure intuition of space) rests solely with the fact that it is used in comprehension to define and compare objective forms of appearances (that is, the "relative and movable space" in which an object appears). Thinking can embed an empirical space within another empirical space, and embed that second empirical space into yet a third, and so on without any known limit. Therefore absolute space signifies nothing other than a capacity for assimilations in apprehension and a capacity for accommodations by differentiations in appearances.

Thus far, Kant has spoken freely of "motion" without providing a fundamental explanation of how we are to understand this term (much as Newton and Descartes did). But if empirical space is merely the formal arrangement in our experience of motion, it is clear that a sharper explanation of "motion" must be provided. Kant provides an explanation that seems, at first encounter, somewhat circular. We will first look at this explanation, and afterwards find out how to escape what seems at first to be a circular argument.

> Motion of a thing is the *change of its outer relationships* to a given space.
>
> I have so far placed the concept of motion at the basis of the concept of matter. For, since I wanted to determine this concept independently of the concept of extension . . . I could allow the common explanation of *motion as change of place* to be used. Now, since the concept of matter is to be explained generally, and therefore as befitting also moving bodies, this definition is no longer sufficient. For the place of any body is a point. . . Now a body can move without changing its place, as the earth in rotating around its axis. But its relationship to outer space still changes thereby . . . Only of a movable, i.e. physical, *point* can one say: motion is always change of place. One could object to this explanation[13]: that inner motion, fermentation for example, is not included; but the thing one calls moving must to that extent be considered as a unity. . . The motion of a thing is not the same as motion in this thing, but here we are concerned only with the former case. But the application of this idea to the second case is then easy [KANT15a: 196 (4: 482-483)].

So we hope. Kant has here given us a *Realerklärung* of "motion of a thing," not "motion in general." This "motion" is more restricted than the full-blown Greek idea of *kinesis*. Hence, "fermentation" as motion is excluded from the given definition. Kant points out that motion as he has explained it here is a special case, i.e. it is motion of bodies; he intends for motion in the *kinesis* sense of the word to have a wider scope. We know something as corporeal motion by knowing of a change in the relationship between the appearance of the thing and some other "given" (that is, observed) empirical space. After several pages of increasingly detailed technical discussion of motion as it applies to corporeal bodies, Kant provides us with an explanation that takes us into the key idea that we are seeking by means of this review.

> The *composition of motion* is the representation of the motion of a point as the same as two or more motions of this same point combined together.

[13] i.e., object to the explanation of motion as change in outer relationships to a given space.

> In phoronomy . . . motion can only be considered as the *description of a space*, in such a way, however, that I attend not solely, as in geometry, to the space described, but also to the time in which, and thus to the speed with which, a point describes the space. Phoronomy is thus the pure doctrine of magnitude (*mathesis*) of motions. The determinate concept of a magnitude is the concept of the generation of the representation of an object through the composition of the homogeneous. Now since nothing is homogeneous with motion except motion in turn, phoronomy is a doctrine of the composition of the motions of one and the same point according to its speed and direction [KANT15a: 202 (4: 489)].

The key idea we are looking for is in here. It is: motion is that which is regarded as the description of a space as this description is developed in the generation of the representation of an object. Now, this is where Kant's earlier explanation runs into the problem of looking circular. If motion is change of outer relationships with respect to some referenced space, but is also only to be considered as the description of a space, it looks like we're being presented with a chicken-and-the-egg paradox.

The escape from the paradox is found by remembering that we have two different "types" of space to consider: the pure intuition of subjective space and empirical space. *Composition* of motion is the representing of a series in subjective time in the synthesis of apprehension, thus as the *comprehension* of multiple concepts of appearances to form one phenomenal object, namely "the motion." (The concept of motion is a concept of an *Unsache*-thing). It thus involves intuitions from multiple moments in subjective time, and each such intuition has a form, namely its spatial *Gestalt*. Motion describes an empirical space, but the pure intuition of space *makes possible the observation of motion* as the magnitude of a difference in appearances between successive moments in time. Armed with this idea, we can leave off from our review of Kant's metaphysic of natural science and return to our discussion of the topological synthesis.

Gestaltung and the Topological Synthesis

If we may judge from what we find in the *Opus Postumum*, Kant found it difficult verging on unaccomplishable to carry on with completing a more detailed explanation of the pure intuition of space (one that would take in "motion" in the general *kinesis* sense). In a way this is something we can understand and for which feel sympathy. The ideas that go into the mathematical topic of topology, or even the idea of topology as a formal topic, had not yet occurred to mathematics in Kant's day. Fortunately, we have history as our ally and will be able to get a little farther along.

We start from descriptions provided by Kant in the judicial Standpoint. There is little to no doubt that Kant saw the pure intuitions of space and time as key links in a seamless transition from metaphysics to physics. Furthermore, *general* physics was seen by Kant as going past dealing merely with what he liked to call "dead matter" to take into its scope *all* natural phenomena, including psychological phenomena.

> Perception can be outer or inner (i.e.) sensation; the latter (in reference to the Object) can be feeling of *Lust* or *Unlust*, that is, one which strives to remove the sensation or to unite it with itself, and issues in appetite or repugnance. Both go to outer or inner experience, hence to the province of physics [KANT10: 145 (22: 500)].

To our modern way of viewing things, this seems an odd and perhaps even controversial stance for Kant to have taken. However, here we need to realize that Kant saw physics not in terms of "corpuscles" or Greek "atoms" but, instead, in terms of something he called "moving powers of matter." "Matter" of an object in *this* context refers to the potential capacity of an object to affect us, and to affect us means to cause a change. A "moving power" is the idea of grounding causality in the *Kraft* of the object. Kant's metaphysic of natural science is a system that presents a radical and fundamental departure from Newtonian physics. It is much, much closer to modern physics' ideas of interactions described in terms of field theories. Electromagnetism and quantum mechanics are both theories of this sort (although the current view favors looking at the physical "picture" in terms of "particle exchanges"; nonetheless, "particle" is a tricky idea in quantum physics, and whatever else physics' "particles" may be, they are not "corpuscles"). Kant probably first came to the idea of moving powers from Leibniz' theory of monads, but a moving power is not a monad nor does it require the idea of a monad. The empirical Self is an object in Nature, and affective perceptions play a fundamental role in the appearance of the Self. Thus, a complete system of physics must deal with them.

> There is . . . in these foundations of natural science a tendency toward physics, i.e. to a system of the moving powers of matter which must be taken from experience, and whose investigation . . . as a system of these powers is called physics. . . In it [physics] natural science represents the concept of matter as the *movable insofar as it has moving power*, and it contains the empirically given moving powers of matter so far as they are thought together in a system (physics) formally and *a priori*.[14] Any physical body can be regarded as a system of moving powers of matter, and what constitutes for such a system its *a priori* conceivability can be embraced under the title of the general-physiological foundations of natural science . . . So, then, the metaphysical, the general-physiological, and, finally, the physical foundations of natural science will, together, represent the system of moving powers of matter as a transition from the metaphysics of nature to physics [KANT10: 51 (22: 189)].

The sensations experienced by the Organized Being come within the scope of this more general view of what is to be addressed by general physics. Sensations clearly are registered

[14] Kant does not mean we know specific moving powers *a priori*. Physics *as a system* requires its supersensible ideas, and these ideas provide the formal and *a priori* factors in physics simply due to the fact that an idea of a supersensible object can never be other than intelligible, cannot be directly given in experience, and is *ipso facto a priori*.

effects in the appearance of the Organized Being, and our model of this Organized Being can place the immediate source of these effects nowhere else than within the Organized Being itself as *noumenal* Subject. Hence,

> . . . sensation (which is the perceiving subject's own effect) is, in fact, nothing other than the moving power determining itself to composition, and the perception of outer objects is only the appearance of the automate of assembly[15] of moving powers themselves affecting the subject.
>
> What thus belongs first of all to physics are the formal differences of the active relationships of the moving powers of matter, which make their Object into an object of experience [KANT10: 121 (22: 384)].

> It is not in the fact that the subject is affected empirically by the Object (*per* receptivity), but rather that it affects itself (*per* spontaneity), that the possibility of the transition from the metaphysical foundations of natural science to physics subsists [KANT10: 121 (22: 405)].

Now, the objective validity for all of this can only be a practical objective validity. Thus it is grounded in the process of formulating experience. Perceptions are not stamped into consciousness but rather are coalesced in the continuity of the aesthetic Idea.

> Whatever is an object of perception (empirically givable) is not, for that very reason, at once an object of experience – for the latter, as a system of perceptions, must be *made*. Now, all outer perceptions are effects of the influence of the moving powers of matter and of the outer Object affecting the subject, and, to that extent, merely appearances; thus, they can be given, as to their formal element, *a priori* [KANT10: 123-124 (22: 408-409)].

> Perception is itself the effect of an act of the moving power of the subject determining for itself *a priori* a representation [KANT10: 161 (22: 439)].

Kant used the word *Gestaltung* for this process of the synthesis of pure intuition in the making of sensible representations. The word does not travel well into an English equivalent, but the connotations of *Gestaltung* include the ideas of formation, forming, construction, shaping, fashioning, creation, and production.

> All our faculty of knowledge subsists in two acts: intuition and concepts; both, as pure (that is, not empirical) representations . . . emerge from the capacity for representation, from *Gestaltung* (*species*[16]) and thinking [KANT10: 183 (22: 419)].

> Positing and perception, spontaneity and receptivity, the objective and subjective relationships, are simultaneous because they are identical as to time, as appearances of how the subject is *affected* – thus are given in the same *actus* and are in progression toward experience as a system of perceptions [KANT10: 132 (22: 466)].

The picture that emerges from this judicial Standpoint is one that is far removed from the wax-tablet or blank-paper picture painted by Aristotle and the British empiricists. Its character is not that of a rapid, crisp presentation but has more of the flavor of a picture coming into focus.

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[15] *der Automatie der Zusammenfügung*. The phrase more or less denotes "self-acting assembly."
[16] Something presented to view, as in a spectacle.

> To *make* an experience (through observation and experiment) is an asymptotic endeavor [KANT10: 253 (21: 93)].

Let us now apply Kant's foregoing remarks to our topological Object, X. In the discussion that follows it is to be understood that although we refer to X and its points *x* as "Objects," these are merely the practical Objects of our theory and are not what is perceived by the Subject as an object. In other words, to describe the synthesis we must adopt a normative convention from which we can make the theory, and it is only from the viewpoint of this convention that X is an Object. Because the logical occurrence of X is prior to experience, X and its points are not themselves to be regarded as direct objects of any possible experience for the Subject. From the "modeler's perspective" of our normative convention, however, we are allowed to "picture" X and its points in terms of somatic activity patterns comprised of neuronal and hormonal signals happening in objective time over some (possibly variable) objective time interval Δt.[1] We are allowed the use of this "picture" as a model to aid our thinking because of the principle of thorough-going reciprocity between the empirical representations of *soma* and the mental representations of *nous*. Whatever information subsists in the noetic representations of the Subject, this same information also subsists in the somatic signals of the Subject's body (which is an object of appearance for us as observers).

Let $x^{(1)}$ be the effective accident of *Existenz* in receptivity represented by somatic signaling events during interval Δt_1. We may take $x^{(1)}$ to constitute a subset T_1 of X in its own right. Now, $x^{(1)}$ also has the significance of being the matter "filling" some empirical (relative) space and, as such, is regarded as that which contains the causality of an effect on the Subject. From the theoretical Standpoint of Kantian applied metaphysics,

> Matter fills a space, not through its mere *Existenz*, but through a *particular moving power* [KANT15a: 210 (4: 497)].

Kant defines a moving power as "the cause of a motion," and in our current context we take "motion" in its most general (*kinesis*) meaning. By "filling" a relative space we mean only that we make a transcendental affirmation that the form of the appearance represented as the somatic signals *embodies* a representation of the effective accident $x^{(1)}$. ("Space-filling" is a Quality in

[1] Δt is the correlate in objective time to the interval between moments in subjective time. Because the marking of a moment in subjective time is an act of reflective judgment, the occurrence of this marking in principle must have at least one characteristic in somatic signaling that stands as somatic correlate to the noetic event. Problematically, then, if neuroscience were able to identify this somatic characteristic then neural activity could stand as a clock for instrumentation of Δt. In the somatic "picture" described here, the role of Δt is analogous to a division made by Malsburg, in which he describes a hypothetical time interval over which binding of neural signals by temporal correlation occurs [*op cit.*, Malsburg, *Neuron*].

Kant's applied metaphysic of Nature). Interpreted on the noetic side of Organized Being, the representation of the moving power of $x^{(1)}$ contains an intensive magnitude of *Lust per se*. This representation signifies the matter for a possible affective perception, i.e. a possible connection to the faculty of pure consciousness in a Relation of matter of Attention.

The moving power of $x^{(1)}$ implicates affection of the process of the synthesis in imagination and the process of reflective judgment. Through the latter is implicated a sensorimotor act of judgment a_1; through the former is implicated an act of imagination s_1. Symbolically,

$$x^{(1)} \rightarrow a_1 \; ; \qquad x^{(1)} \rightarrow s_1 \; .$$

Now, an act is the form of an activity, the matter of which is action. This occurrence contains a co-determination of the Organized Being and its environment which implicates an effect due to moving powers of external environmental objects, hence a receptive action r_1. Taken together, these actions implicate, in the next interval Δt_2, another effective accident. Letting $\langle a_1, s_1, r_1 \rangle$ denote the sensuous circumstance, we denote the generation of this new effective accident arising from the sensorimotor activity and imaginative synthesis as

$$\langle a_1, s_1, r_1 \rangle \rightarrow x^{(2)}.$$

We now look at the possible *forms* in which this effective accident may be represented during interval Δt_2. One possibility is, of course, merely $T_2 = x^{(2)}$. Another is that which is common to both $x^{(1)}$ and $x^{(2)}$, i.e. $T_3 = T_1 \cap T_2$.[2] Third and fourth possibilities consist of differences between $x^{(1)}$ and $x^{(2)}$, i.e. $T_4 = T_2 - T_1$ and $T_5 = T_1 - T_2$.[3] Here we may note that $T_4 \subseteq T_2$ and $T_5 \subseteq T_1$. Finally, owing to the interplay of determining judgment and the synthesis of imagination, we note that it is possible for T_2 to contain T_1 in its entirety, i.e. for $T_2 = (x^{(2)} - x^{(1)}) \cup T_1$.[4]

The topological synthesis of the pure intuition of space is nothing other than the formation (*Gestaltung*) of such sets of *possible* effective accidents. Which of these end up in *actual* intuition is not the business of the pure intuition of space. That determination is tasked to the process of reflective judgment, which acts in accordance with the principle of the formal *Zweckmäßigkeit* of Nature. The task of the pure intuition of space is to supply the determinable possibilities as "givable" (*dabile*) in sensibility. In terms of our somatic picture, these T-sets are interpreted as

[2] Mathematically, it is possible for T_1 and T_2 to share no common factors, and in this case the intersect is null. We interpret this as an accident that is *impossible*, i.e. *not givable* in sensibility.
[3] In mathematical notation, T_4 and T_5 are called set differences. If Y = {a, b, c} and Z = {a, b, d, e}, then we have Y – Z = {c} and Z – Y = {d, e}. Note that set difference is not generally commutative. If Y = Z then both set differences are null, and these, too, are regarded as impossible accidents.
[4] $x^{(2)} - x^{(1)}$ denotes that $x^{(1)}$ negates what is common with it in $x^{(2)}$. This is merely a notational convention.

possible bindings in neural activity, and so we could say that these sets subsist in the "binding codes" of Damasio's model (taking Damasio's hypothesis as a merely problematic prototype used as an analog for understanding the pure intuition of space).

A few remarks about the process just illustrated are in order. First, it is to be noted that as pure knowledge *a priori*, the pure intuition of space does not *present* a topology but, rather, *builds* a topological empirical representation of space *as a system*. That the process just explained satisfies the requirements of inclusion of intersects and unions of the various T_i called for in the mathematical definition of a topological space is obvious.

As to the mathematician's requirement that the "null set" belong to T, this idea requires some additional consideration. First, the inclusion of the null set required in the formalism of mathematics is partly due to the old complaint by symbolic logic that Aristotelian logic had an "existential import," e.g. "all griffins are fierce" implies "griffins exist." The logical possibility of a null result for some set operations in our topological synthesis does have a *practical* interpretation. If for two possible sets T_a and T_b we have no common intersect, the proper interpretation of this condition is that T_a and T_b represent *distinct empirical spaces* (which is allowed for empirical relative spaces in the Critical Philosophy). This condition is none other than *the ground for the possibility that the practical sensorimotor spaces of the infant are not co-ordinated in the early stages of life*.

We may furthermore note there is nothing inherent in the idea that every intuition is a singular representation which says the Organized Being cannot have more than one intuition marked out in any given moment in time. Sensibility contains a manifold, and multiple intuitions can constitute a part of this manifold. Their presence merely means that we can attend to multiple matters of Attention.

Third, the condition of continuity in the aesthetic Idea links representation in intervals Δt_1 and Δt_2. We may look at this continuity as, in a manner of speaking, a *tracing* of motion between points *x* in empirical space. Motion "is the description of a (relative) space" and the condition of sensuous continuity grounds this possibility in the syntheses of imagination and apprehension. Our coming-to-understand objective spaces through general *Beurtheilung* is founded upon this condition of continuity and the "trace-ability" of points in empirical spaces.

Fourth, this same interplay of processes in the faculty of knowledge provides a transcendental ground for exploring the *physical* question of how activity patterns can come to be associated in, e.g., Damasio's hypothesis. On purely empirical grounds Damasio's hypothesis that binding codes arise from convergence zones and "associate" sensory "fragments" as entities and events is an idea *ex post facto* from scientific experience. Neither classical empiricism nor classical rationalism was ever able to offer an explanation for the "how" of the phenomenon of associations. Positivism could allow itself to do nothing other than say, "Well, look, it just happens," and forbade itself by its own pseudo-metaphysical prejudices from further exploring the issue – thereby denying the possibility of constructing a proper scientific doctrine as a system. Scientific materialism (or, rather, the method of scientific materialism) could pursue the question on a strictly reductionist basis but, in subordinating the phenomenon of mind to that of body (and subsequently trying, as in behaviorism, to eliminate the former), denied itself the necessary link for a systematic anasynthesis for re-integrating its reductionist findings to obtain a whole, which is possible only by a synthesis *a parte priori*. It is perfectly true that the Critical Philosophy comes ultimately to a stopping point, so far as further reductionism past *noumena* is concerned, but this stopping point is *at the boundary of objective validity*, past which we enter the realm of transcendent opinion. The advantage it brings is "knowing when to stop" based on *a priori acroamata* of the Critique of pure reason.

Fifth, the process of topological synthesis, as it is explained here, has the formal structure to match up with Piaget's structure of adaptation. We can regard the "tracing" of a closed loop in empirical space as the formal counterpart in receptivity to Piaget's closed cycles of adaptation in assimilation and accommodation [PIAG1: 5-6]. Indeed, "trace closure" in sensory space by means of the general *Beurtheilung* of the faculty of knowledge in an Organized Being is nothing less than the rational ground of the possibility for Piaget's empirical circular reactions and for the possibility of sensorimotor schemes *in concreto*.

Finally, all empirical (relative) spaces are limited, and in this sense can be regarded as sensory "fields." However, Kant's absolute space subsists in the knowledge *a priori* of the rules of the pure intuition of space by which empirical spaces are constructed. We cannot mark *a priori* a condition of limitation on this absolute space – that is, we know of no requirement by which the products of the topological synthesis are necessarily finitely numerable. Kantian space "contains an infinity in itself" (to quote Kant), and *practically* this means nothing other than that we know of no finite limitation in its capacity to synthesize empirical spatial topology. "Infinity" is for philosophy a "becoming" and we have no idea of a bound for what can come out of the synthesis of imagination, which the pure intuition of space serves in receptivity.

§ 7. Postscript: The Aesthetic of the Pure Intuition of Time

Before ending this chapter a few words are in order regarding the pure intuition of inner sense. We have seen that subjective space and subjective time are inseparably involved in the synthesis of sensibility. It is worthwhile here to briefly go over Kant's theoretical description of time.

Having seen the variety of conflicting historical ideas of space earlier in this chapter, the reader perhaps will not be surprised to find that similar controversies embroil the idea of time. I postpone the review of these controversies until Chapter 21 and here merely point out that these points of contention over the question, "What is time?" are, like those of space, centered on *objective* time. One question of ancient standing, still with us today, is: Did time have a beginning? As you might easily imagine, this question does not admit to an easy and straight-forward answer. It has been discussed and debated by Aristotle, St. Augustine, Newton, Leibniz, and numerous others as well. The "nature of time" is still something of a contentious issue in physics today.[5]

That young children have no innate knowledge of objective time will come as no surprise to most of us, since all of us can probably remember being *taught* how to "tell time." Of course, personal anecdotes are not scientific evidence, but that this is so was proven beyond doubt by Piaget and others, and is well documented by psychology. Piaget provides detailed documentation of how the child's conception of objective time develops [PIAG6].

Kant's breakthrough came from recognizing the real division between time as an object and time as the pure intuition of inner sense. We briefly discussed this in Chapter 6, and have been using subjective time throughout this treatise. Kant's theory, like his theory of space, has been widely misunderstood, and again we can blame Kant himself in part for this because he did not explicitly *say*, "Look, there is subjective time as intuition and objective time as object, and they are not the same" in *Critique of Pure Reason*. It will perhaps be obvious to you by now that the pure intuition of subjective time is the ground of the possibility of developing ideas of objective time. In *Critique of Pure Reason* Kant tells us,

> This *a priori* necessity [of subjective time] also grounds the possibility of apodictic first principles of relationships of time, or axioms of time in general. It has only one dimension; different times are not conjoint but after one another (just as different spaces are not after one another but conjoint). These first principles could not be deduced from experience, for this would give neither strict universality nor apodictic certainty. We would only be able to say: This is what common perception teaches, but not: This is how matters must stand. These first principles are valid as rules under which generally experiences are possible at all, and instruct us prior to them, not through it [KANT1a: 179 (B: 47)].

Neither of these two "first principles" of *objective* time presently seem to be in serious doubt, although many other ideas of objective time have been and are still seriously contentious. Feynman, for example, theorized that antimatter is "really" ordinary matter moving *backwards* in

[5] Recently *Scientific American* devoted an entire special issue to some of the issues and problems the question of time presents to present-day physics (Vol. 287, no. 3, Sept. 2002).

time, from "the future" back into "the present" and proceeding thus "into the past" – and this is one of the elements of the theory for which he won the Nobel Prize.

In many ways subjective time and subjective space are what logicians call "duals" of one another, i.e. that which can be said of one of them can, by exchange of particular operators or ideas, also be said of the other. For example, in electrical engineering two circuits are said to be "duals" of one another if they are both described by exactly the same form of mathematical equation except for an interchange of voltage for current and conductance for resistance. Kant gave an example of duality for objective space and objective time in the quote above. The basis of the "duality" of subjective space and subjective time is found in that the former is the pure intuition of outer sense, the latter the pure intuition of inner sense. Of duals one might say that they are really different but metaphorically the same. (We will see more of this in Chapter 21).

As was the case for subjective space, the objective validity of subjective time is only a practical objective validity, and the proper Standpoint to be taken for subjective time is the judicial Standpoint. From the theoretical Standpoint subjective time, like subjective space, is not a "thing" but rather a capability. The theoretical characterization of this capability is described under the four titles of representation in general as follows. Under the general heading of Relation, Kant tells us

> Time is not an empirical concept that is somehow deduced from an experience. For coexistence [*Zugleichsein*] or succession would not themselves come into perception if the representation of time did not ground them *a priori*. Only under its presupposition can one represent: that several things be at one and the same time (conjointly) or in different times (after one another) [KANT1a: 178 (B: 46)].

In his *Prolegomena* Kant pointed out that *concepts*, once represented, cannot be properly arranged temporally in their re-presentation by the synthesis of imagination unless it was possible for this temporal ordering to be pre-fixed in intuition. We discussed this earlier in this treatise when we dealt with the transcendental schemata and their relationship to the categories of understanding. We cannot be given this ordering through the senses because "time" is not an object of the senses. The three *modi* of persistence in time, succession in time, and coexistence in time are characteristics of Relation givable (*dabile*) in sensibility only *a priori* through the pure intuition of inner sense. Time is internal Relation in representations, i.e. Relation in inner sense, and this is logically categorical.

Under the title of Modality we have,

> Time is a necessary representation that grounds all intuitions. In regard to appearances in general one cannot remove time, though one can very well take the appearance away from time. Time is therefore given *a priori*. In it alone is all actuality of appearances possible. The latter could altogether be omitted, but time itself (the universal condition of their possibility) cannot be

removed. [KANT1a: 178-179 (B: 46)].

All appearances are said to "fill time," and in this sense one is tempted to regard time as some sort of "container." But this is not a particularly good simile, and it is better to say that time is the capacity to order presentations in sensibility in the manner which we, as human beings, perceive Nature. Subjective time viewed as matter of the form of the manifold of appearances provides the capacity to organize Nature insofar as this organization is with respect to temporal appearances, i.e. to properly "put in their places" (in a manner of speaking) concepts in the re-presentation of their intuitions. Time is a determining factor in Modality, hence logically necessary.

In terms of Quantity,

> Time is no discursive or, as one calls it, general concept, but a pure form of sensuous intuition. Different times are only parts of one and the same time. Their representation, however, which can only be given through a single object, is an intuition. Further, the proposition that different times cannot be conjoint cannot be derived from a general concept. The proposition is synthetic, and cannot arise from concepts alone. It is therefore immediately contained in the intuition and representation of time [KANT1a: 179 (B: 47)].

This characteristic of subjective time speaks to its *integrative capacity*. Moments *in* time are merely the boundary marks that define "different times" in appearances. Time is universal.

Finally, in terms of Quality,

> The infinitude of time means nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation *time* must therefore be given as unlimited. But where the parts themselves and every magnitude of an object can be determinately represented only through limitation, there the entire representation cannot be given through concepts (for they contain only partial representations), but immediate intuition must ground them [KANT1a: 179 (B: 47-48)].

Just as the reality of an object must presuppose an "All-of-Reality" as its substratum, a moment *in* time must presuppose an "All-of-Time" as its substratum. The presentational capacity of the pure intuition of time works through particular limitations, just as the presentation of an object in space works through limitations on an "All-of-Space." This Quality of subjective time is the idea of subcontrarity in our general 2LAR of representation. Time is judged logically infinite.

It is interesting to compare the four titles of logical judgment for subjective space and time. They differ only in their form of *nexus*, that is, Relation. Space is characterized as universal and disjunctive (transitive); time as universal and categorical (internal Relation). The interplay of this difference is what makes possible anasynthetic presentations of anticipations of reasoning in intuition through the synthesis of comprehension in Kantian Logic (i.e. the hypothetical is obtained as the synthesis of the categorical and the disjunctive). The *Gestaltung* of pure intuition in general requires both space and time working together to make experience possible.